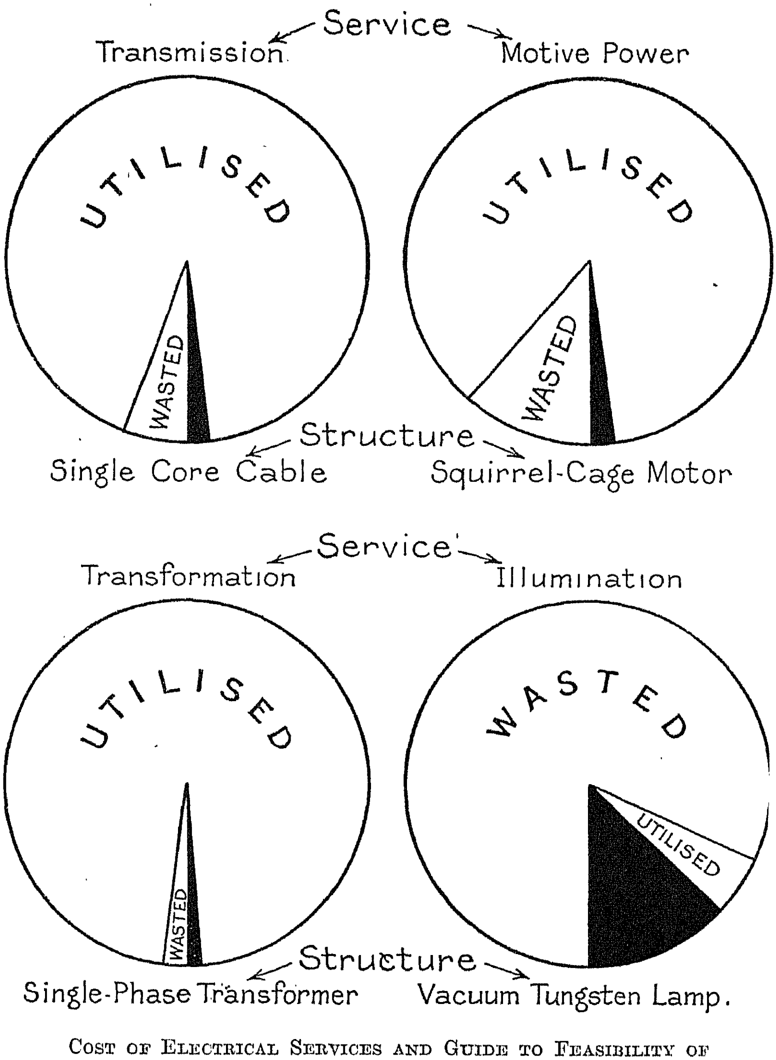
ELECTRICAL ENGINEERING ECONOMICS



COST OF ELECTRICAL SERVICES AND GUIDE TO FEASIBILITY OF ECONOMIC CHOICE.

Area of each circle represents total cost of the service named. This consists of a black portion giving cost of structure, and a white portion giving cost of energy entering structure. The latter is divided into cost of energy utilised and cost of energy wasted in heat. (See table and data on p. 102.)

[*Frontispiece.*

ELECTRICAL ENGINEERING ECONOMICS

A STUDY OF THE ECONOMIC USE AND SUPPLY OF ELECTRICITY

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PREFACE

There is no need to stress the importance of a knowledge of economics to practising engineers at the present day. Many of the decisions which have to be made concern costs quite as much as performances, and items such as interest and depreciation have to be considered, as well as strength of materials or magnetism and electricity. In fact, to be fully qualified the engineer can no longer be content with a knowledge of the properties of matter—he must also know something of the reactions of his fellow creatures. But even when recognising this fact, engineers (like the economists themselves) have been prone to dwell almost exclusively on the economics of production, with the result that most of the books on this subject have dealt only with production problems—costing, price fixing, and works management economy—themes which are not touched upon in the present volume.

There would therefore appear to be a real need for a book dealing more particularly with what may be called consumption economics, especially as it is in the consumption that the greatest waste occurs; and clearly it is no use taking great trouble to save a fraction in the power house, only to squander it wastefully in the workshop or the home. The manufacturer *per se* is naturally concerned only with efficient production without any consideration for the efficient use of what has been produced, but the engineer has to use plant as well as make it, and it is in this sphere of economic utilisation that he is apt to fall short. The physical characteristics (such as strength and output) of a proposed installation are understood fully and exactly, but when it comes to the *economy* of the performance he is too often content with a categorical "cheap" or "dear." It must not be forgotten that cheapness, like efficiency, is a matter of degree, and any tendency, however good in itself, has an economic limit beyond which it should not be pushed.

One of the objects of the present book is, then, to attempt, in a few cases occurring to the user of electrical plant and energy, to answer the question "what is cheapness?" in any real sense of that much-abused word. It is intended to enable

engineers and others to find what precisely is the total cost of some given electrical service and thus to compare alternatives. In this way it is hoped finally to pave the way towards a complete set of economic criteria for the design and choice of engineering plant.

In brief, the aim of the book is to give to electrical engineers and students a plain account of such elementary economics as most nearly concerns them, together with its application to certain engineering problems. It is in three roughly equal parts, the first of which deals with general principles. This part makes no claim to originality, the definitions and explanations being those of the standard economists, and the formulæ being such as can be found in most engineering pocket-books. At the same time, an attempt is made to re-state the classical treatment of productivity on lines at once more intelligible and more useful to engineers.

The second part of the book deals with the general problem of economic choice, to which the name "consumption economics" can particularly be applied. It may be thought that to speak of consumption economics as a separate branch is a mistake, since every step in the process which makes natural resources minister to human wants consists of consuming one thing to produce another. But although every process involves both things, it can be looked at from either end—that of the producer who is paid for the service, and that of the consumer who pays. Moreover, the data is different in the two cases, since the former has a fixed equipment or capital and desires the maximum profit, whereas the latter has a fixed need or service and desires the minimum expense. The distinction is therefore a useful one, and processes regarded from the consumption end constitute a separate and little treated type of problem.

Owing to the large number of possible variables, problems of this kind have usually been treated in strict isolation, and very little attempt has been made to link them up with other cases so as to build up a complete theory and practice of economic choice. The possibility of a general solution of such problems by means of the conception of "service-price" developed in Part II. is a first step in this direction.

The third part of the book deals with some of the economic problems connected with electricity supply. Sometimes these are also cases of economic choice of plant, as in the chapters dealing with power factor, but for the most part they are concerned with certain economic aspects of load factor, tariffs,

generation and distribution. The two-part diagram illustrated in the folding plates is an endeavour to express exact and complex economic data in a vivid and comprehensible manner; and it is hoped that it will be useful both to the supply engineer and also to the non-technical consumer who wishes to understand exactly what makes up the cost of his supply. Similar remarks apply to Figs. 23 and 25 in the selection of phase improvement plant, and like almost all the figures here employed these are necessarily original since they illustrate matter not previously dealt with in an engineering text book.

In most of the chapters the principles enunciated are capable of immediate application to electrical engineering problems, and a number of worked examples and illustrations are given. These appear during or at the end of the chapter in question, and very great trouble has been taken to make the data employed thoroughly representative of present-day figures and practice.

My thanks are due to Mr. F. Gill for a number of helpful suggestions in connection with the chapters on depreciation, to my wife for a very large amount of assistance, and to the following for permission to reproduce extracts from my original papers (in each case detailed reference is made in the text where the extract occurs) :—

The Institution of Electrical Engineers; World Power; The Electrician; The Electrical Review; The Electrical Times.

D. J. B.

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PART I

GENERAL PRINCIPLES

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CHAPTER I

CAPITAL, INTEREST, AND SINKING FUNDS

Capital.—When men work, whether individually or collectively, there is usually involved a certain reserve or margin of wealth, useful or necessary to the operation, and this " head " of working is called capital.

Thus when the results of the labour are not obtainable for a period of time, this wealth must include amongst other things sufficient spare subsistence to keep the labourer alive until the end of the period ; and such a reserve can be regarded as the basic and most elementary form of capital. With this might be grouped the raw material, seed, stock, or other " trading assets," the whole being then designated fluid or circulating capital.

In the second place, most modern operations involve the use of tools and equipment of some sort, and this type of capital (referred to hereafter as " plant " or " structure ") is the most general form, and the one chiefly dealt with in engineering problems. It can be regarded as being generated from the first kind when men, before engaging on any enterprise, make the tools they wish to use, whilst living in the meantime on their store of " spare subsistence."

A third variety is the non-material structure implied in such words as " business organisation," " goodwill," and the like. A large undertaking, like a river, must develop smooth and suitable channels for its flow, and this may involve not only offices and agencies, but publicity and advertising, all of it a conversion of the original " spare subsistence " into an elaborate equipment for the better production of wealth. This is not only the least tangible, but also the least obviously useful form of capital, since the wealth it helps to produce, although having an economic value in satisfying human desires, may have no intrinsic worth if the desire is an artificial one created

by the advertisements. Capital which is sunk in assets of the two kinds just mentioned is frequently called fixed (as distinct from circulating) capital.

It will be seen that the stock of goods which is called capital differs from other goods, not by what it consists of, but by the way in which it is used. A motor car can be used for business purposes or purely for idle pleasure, a diamond may be employed as a glass-cutter or as an ornament; in the former case the goods would be classed as capital, and in the latter case not. From its definition as "accumulated wealth used in producing more" the distinguishing feature of capital is, therefore, its productivity or power of economising labour. In wealth production there are usually alternative paths open for the attainment of the desired end, and capital derives its intrinsic value from the fact that the longer or more elaborate path is usually the more efficient. Labour goes further when lubricated by capital, or—to vary the metaphor—capital is the tool which lengthens man's arm and enables him to encompass many things not otherwise within his reach.

Thus although it is usually possible, in an undeveloped and thinly populated country, to produce wealth without capital, such a livelihood from the immediate fruits of the earth and beasts of the forest is usually more precarious and laboursome than husbandry and shepherding. The latter is therefore in the long run more productive per unit of labour, but it necessitates capital—stock and seed corn, enclosures and spare subsistence, even laws, governors and, ultimately, rates and taxes, since the governmental machine itself is to some extent a structure for the better production of wealth.

Capital is therefore, if not as old as the hills, at least as old as the cultivated valleys,* and must not be confused with capitalism—that is to say, the particular present-day system of ownership and concentrated control of capital, which is of comparatively recent origin. Another thing which is recent in most industries, and which goes hand-in-hand with the present system of ownership, is the use of very elaborate and expensive capital instruments, even for the manufacture of

* It is difficult to say, even very roughly, how old this is. There seems reason to believe that racially man has existed in very much his present form for about a million years, but it was only some five to ten thousand years ago that the cultivation of corn, the enclosing of herds and the use of simple vessels and implements first gave him leisure to be more than an animal. Then it was that capital first emerged, and then it was that civilisation can be said to have commenced.

small and simple objects. If any long-established industry connected with food or clothing is examined, *e.g.*, shoemaking, it will be found that simple tools such as hammer and last have been employed from the beginning, but it is a characteristic modern development to employ a whole series of elaborate machines in manufacturing a single shoe. Industries of more recent origin, such as the electrical engineering one, have been highly capitalised from the first, since many of the processes involved are necessarily intricate and do not lend themselves to small-scale hand operation.

Land or Nature.—Since man cannot make something out of nothing, he can only *create* what are called "utilities," and the business of producing wealth or economic values consists in converting materials into more useful forms, or intensifying useful natural processes. The only essentials in wealth production are therefore labour and some degree of access to natural materials and resources; given a few acres of earth, lake or ocean to live on, and to fish, hunt or dig in, man can live. The natural fertility of land being the chief and type of these resources, the word "land" is frequently used in economics to cover all the sources or raw materials of wealth production, whether in earth, air or water, mines, fountains, fisheries, etc. As will be seen below, in engineering problems the word "land" tends to have other meanings and a different emphasis, and the word "nature" or "natural resources" is preferable.

Whichever word is used, there are certain features which distinguish this "nature" element from all others which take part in production. In the first place it is essential, and not merely helpful like capital or initiative. Secondly, it is definitely limited in extent, area or what not, and so has no "supply price"—that is to say, however much the demand for it (and therefore the price) goes up, no additional amount will be thereby called into existence.

Although described as the "free" gift of nature, owing nothing to man's labours, this element has a price (at least in well-populated countries) called "economic rent"; but this is a monopoly price due to its scarcity, instead of reflecting, like other prices, the cost of production. Thus if the population were by some means reduced to one-tenth, rent charges might be enormously reduced although the labour costs of manufacturing an article were the same or more. Economic

rent can therefore be considered to have arisen because the population has outgrown the number of available best plots of ground, etc.

Unfortunately, useful as the above distinction is as a mental help in studying the subject, it is extremely difficult in practice to say where to draw the line which divides the inextensible gifts of nature from the more flexible additions of man. Natural resources by themselves furnish but an indifferent and precarious livelihood, and the gradual increase of population has not only intensified those natural differential advantages which issue in economic rent, but has also necessitated building thereon a vast artificial superstructure—the trappings by which nature is "harnessed." By an immense accumulation of past labours, skill and saving mankind has "reclaimed" the land and water, brought them together or kept them apart, intensively cultivated them, erected buildings and bridges over them, enclosed, policed, linked up and generally increased their productivity. Thus it comes about that most portions of the earth's surface which best lend themselves to the possibility of this sort of development have now been parcelled out and worked up into cities and vineyards, docks and empires, having enormously enhanced economic values, and (although the supply of suitable land values is relatively inelastic) this process is still going on.

It will be seen that there is a continual tendency for nature to be parcelled out into private hands and to be "worked up" into capital,* and the dividing line is a faint one. There is, therefore, little to be gained here in splitting up actual rent charges into economic rent and interest rent, or in treating land as a thing apart. A more useful conception is to regard all the productive agents except labour or energy as constituting a species of structure or working equipment (loosely called capital), and costing so much per annum in interest, rent, rates, etc.

In engineering calculations only a small part of this equipment is the element "nature," whilst the remainder (including most of the productive attributes of land as we know it, in a civilised country and near centres of population) can be comprised under the definition of accumulated wealth used in

* Fortunately this is not the whole story, for much of this tends in turn to become socialised capital such as the roads, cities, knowledge and security which we call "civilisation," and thus become re-distributed to the general public. See the concluding paragraph below and also compare the quasi-rents due to new inventions, etc. (p. 75).

producing more, and can therefore be strictly classified as capital. Moreover, in the calculations here involved, this classification will be useful in emphasising the general interchangeability of different types of capital. Thus a new undertaking may be faced with the choice between a convenient site in the centre of the load at a high rent and rate, and a less convenient but cheaper one requiring expenditure in other capital directions, such as mains, sidings, water supply, etc. Or the choice may lie between an extension on the street level at a high ground rent as an alternative to a tall, expensive building having a small floor space.

Summing up, it may be stated that while labour acting upon natural resources is the actual reagent in the production of wealth, capital and initiative act as catalytic agents in assisting the reaction. But by no means all the accumulated facilities which assist production appear as capital or have to be paid for, since most of the past workers have bequeathed their savings free to the general store. No one in a civilised country works without tools or without immense help from his predecessors in accumulated knowledge, skill and traditions, as well as in more material assets. Improvements which are general and widespread represent no exchange value or wealth in the economic sense, and the problem of paying for capital is merely that of paying for the differential advantages—the extra skill or machinery not yet become the common property of all workers.

In considering the cost of capital two points must be noted. Capital, particularly of the dimensions now employed, is rarely the property of one man or of the group of men working on the enterprise in question. It has therefore to be *borrowed* from those who have more of it than they need for their own use.* In the second place (except for some qualities of land), capital is rarely permanent, and so has to be periodically *renewed*. These two points are considered in turn in the remainder of this and in the next chapter.

Rent and Interest.—When goods (or money for the purchase of goods) are lent by one person to another it is customary

* In the discussion which follows it will always be assumed that the borrowing actually takes place. When a man employs his own property as capital it must be imagined that he borrows it from himself, and the profits which he extracts from his business should be sufficient to pay him a rate of interest on the sum as high as he could get elsewhere with the same security, plus a salary for whatever managerial work he performs.

for the lender to make a charge for their use. When the goods are in the form of land, buildings, or other fixed articles, the charge is called rent or hire, but in the case of money it is called interest. It was seen above that the distinction is often misleading, and as money is much the most usual form for the loan to take, and as other sorts of property can be valued in money, it will be convenient to consider all loans as being in this form, and to consider all hire charges as being typified by interest on money.

Interest, *i.e.*, the price paid for the use of a loan, like the price of anything else, is chiefly the result of the interaction of supply and demand, and it is therefore determined by the relative eagerness of lenders and borrowers. With regard to the former, as lending implies some degree of doing without or postponing the satisfaction of personal desires, it is not surprising that it is usually less popular than borrowing; and that being so, it is inevitable for a charge to be made, as otherwise there would be many more borrowers than lenders. The effort of will involved in refraining from the immediate enjoyment of a sum of money may not be great in the case of wealthy persons, but as their savings alone are not sufficient to supply the whole demand, interest has to be high enough to induce sufficient poorer people to save, and the price is governed by these "marginal" cases.

With regard to the borrower, it may be said that loans are chiefly used either to tide over temporary difficulties or else as capital. In the Middle Ages the former use predominated, and it seemed unreasonable and even wicked to make a charge (particularly a time-rate charge) for the emergency use of "barren" metal. Nowadays, with a few exceptions such as war loans, the provision of capital is the chief object of borrowing. And as labour is more productive the greater the capital it works with, some sort of charge becomes reasonable and even necessary, if only to compensate those who are working with less.

Even if capital were not the principal use of loans it would still exercise a determining influence on the price paid, since a man who borrows land or money even for immediate personal gratification has to pay the competitive price for it, governed by its alternative productivity. Thus the interest paid has very little to do with the object of the loan (except in so far as it affects the security), being governed (on the demand side) by the "marginal" productivity of fresh increments of capital,

and (on the supply side) by the quantity of capital available or the willingness of lenders to lend.

In both the supply and the demand aspects of borrowing which have been considered above, the effort required and the service rendered can each be considered as proportional both to the size of the loan (called the principal) and to the time for which it is outstanding. The price paid, or total interest (I), is therefore proportional to both of these elements. A further point to notice in connection with this price is that the lender is commonly considered to perform other functions beyond those of merely "doing without," since he presumably exercises some skill in the placing of his investment, frequently takes some risks, and occasionally performs managerial functions. Payment for these latter services is, of course, more speculative and liable to soar much higher or be absent altogether. It can be roughly grouped under the heading "profits," although the modern tendency is to separate the items out.*

An actual interest payment may therefore include :—

Payment for use of loan—"pure interest."
Payment for risk } "profits."
Payment for administrative skill }

These latter items are absent from perfectly secure investments, and when present their effect is to raise the rate of interest without altering its general character or method of payment.

A final point is, that in speaking of the price of loans as dependent upon the eagerness of borrowers to borrow and the willingness of lenders to lend, it must not be forgotten that the operation takes time. However willing the lenders may be, or however high the rate of interest offered, there is a limit to the amount of money immediately available. All commodities take time to produce, so that (as was seen with many articles during the war) a sudden demand which outstrips the normal rate of supply raises the price far beyond the cost of production. In the case of money loans, the "commodity" is an extremely fluid one, so that very large issues of new capital are often immediately subscribed by the transference of credit from other sources; but the total amount available is obviously limited,

* Thus in a large concern most or all of the management is performed by a salaried staff, whilst the determination of risks and even the placing of investments is often deputed to brokers and investment agencies, *i.e.*, in each case the work is done by disinterested experts rather than by the actual capital owners.

and in the long run borrowing can only be met by saving—which takes time. Thus in times of money shortage the rate of interest may temporarily go far beyond the permanent "cost of production."

In the case of capital sunk in physical assets not easily resaleable the mobility is less, and hence the time element is still more important. At a time when the average rate of interest is 5 per cent., the interest temporarily paid on particular items of capital may be 10 per cent. or zero, owing to particular utility or uselessness. After a time more capital flows in or fails to flow in—the shares are sold at a premium or a discount, and the rate paid to the newcomers tends again to the average. For this reason many economists sharply distinguish between what are called short periods and long periods. During short periods, special demands for capital result in special rates being paid, and in such cases this scarcity price partakes of much of the nature of economic rent, and is often called quasi-rent.

Loan Obligations.—With the change in the uses to which loans are generally put has come a change in the general attitude to them. A loan, except between friends, is no longer a temporary "obligation" of one man to another, with or without interest over the intervening period, it is rather an arrangement between two people, one of whom, say, has £100 and would rather have £5 a year, and the other who has £5 a year and would rather have the £100. So long as the two continue to be of the same mind as before there is no more obligation for the second to give up the £100 than there is for the first to give up his £5 a year; that is to say, so long as the loan continues to be acknowledged and interest paid, there is no reason why it should not go on for ever. This gradual disappearance of the obligation to re-pay with the corresponding emergence of the vital importance of interest can be seen reflected in English law in the restrictions which are put upon the repayment of loans by companies. Thus the terms "debtor" and "creditor" are losing any moral significance, and are being merged into the terms "undertaking" and "shareholder," as representing two parties with reciprocal interests and responsibilities.

Two things should be said about this—although morally the debtor and creditor are now considered to be on the same level, their responsibilities are not quite the same. Since the creditor is due to receive either interest or principal, he is at the mercy

of the debtor should the latter at any time default. Thus it is usual when the creditor passes the money which constitutes the loan over to the debtor, for the latter to pass over in exchange some token of value or credit which will serve as a hostage in case of default. The second point to notice is that when the loan is expended in goods of some sort, as, of course, it will be, these goods become consumed; and although an improved wealth-producing machine (or capital) results, this will in time wear out or become obsolete. Thus, although there is no obligation to repay a loan provided the interest is continued, there is an obligation to replace such goods as become consumed, *i.e.*, to make good all depreciation so as to keep the value of the credit up to its original mark.

Interest Formulæ.*—It will be seen from the foregoing that most loans bear interest which is proportional both to the original size of the loan, called the principal (P) and to the period or length of life (L),† so that a *rate* of interest is both a time rate and a percentage (of principal) rate—so much per cent. per annum. In the formulæ which follow, the rate of interest per annum will, however, be taken per *unit* principal rather than per *cent.* principal, as this simplifies the expressions considerably. Thus with interest at 5 per cent. per annum the rate appearing in the formulæ will be 0·05, and will be denoted by i.

It will be clear from this proportionality, that interest is being continuously generated all the time that a loan is outstanding; but if the principal can be regarded as remaining constant for a year, or if (what is the same thing) the interest can be regarded as being distributed to the lender as it is generated, or else stored inertly for him, then the interest charge for that year can be very simply expressed as Pi, and the amount (principal plus interest) at the end of the year, $A = P(1 + i)$. This is called "simple interest," and making the same assumptions as to constancy, etc., the total interest for L years,

$$I = PiL \qquad (1)$$

and the total amount at the end of L years,

$$A = P(1 + iL) \qquad (2)$$

* The more important equations are numbered and are summarised in the Appendix, p. 291.

† Unless otherwise stated it will be assumed that P and A are in £, and L is in years.

It will be noted that the amounts at the end of successive years form a series in arithmetical progression, since the same quantity (Pi) is added on each year.

In fact, it is not practicable to distribute the interest exactly as it is generated, nor is it usual to store it inertly in a strong box, but rather to re-invest it as soon as possible, so that it also may earn interest. In such cases another method of computation, called compounding, must be employed, and the interest is then called compound interest. The extreme example of compound interest would be where the interest was continually added to the principal as it became generated, the whole then earning interest from that instant onwards. As an example of this latter it might be supposed that the addition was made at the end of each day, so that the interest for the first day, namely, $P \times \frac{i}{365}$, became added to the principal P, giving a new principal $P\left(1 + \frac{i}{365}\right)$ for the second day. This would earn interest $P\left(1 + \frac{i}{365}\right) \times \frac{i}{365}$, and become in turn $P\left(1 + \frac{i}{365}\right)^2$ as the third day's principal; so that the amount at the end of the year would be $P\left(1 + \frac{i}{365}\right)^{365}$, and after L years the amount would be $P\left(1 + \frac{i}{365}\right)^{365\,L}$.

The period of calculation (one day in the above case) is called the compounding interval, and in the extreme case, when this is made infinitely small, the amount after L years, $A = Pe^{Li}$, where e is the number 2·71828, the base of Naperian logarithms. This is sometimes called "true compound interest," or the "law of organic growth," and is met with in a large number of physical phenomena in which the rate of accretion to (or diminution from) an object is proportional to the instantaneous size of the object. Another way of putting it is to say that the number e (which is the limiting value of $\left(1 + \frac{1}{n}\right)^n$ when n approaches infinity) has the property that the rate of change of the function e^x is at all times equal to the value of the function.

Needless to say, it is not practicable to make the interest up every few minutes, nor would it be fair to calculate it in this extreme fashion, since it could not be profitably re-invested at

this rate; it is therefore rare for the addition to be made more often than twice or four times a year. In the case of interest compounded annually and added to the principal at the end of each year (the whole earning interest henceforward), the amounts at the end of successive years will form a series in geometrical progression, since they are multiplied by the same quantity $(1 + i)$ each year:—

	Principal at Commencement.	Interest Earned.	Amount at Termination.
First year .	P	Pi	$P(1 + i)$
Second year .	$P(1 + i)$	$P(1 + i)\,i$	$P(1 + i)(1 + i)$ $= P(1 + i)^2$
Third year .	$P(1 + i)^2$	$P(1 + i)^2\,i$	$P(1 + i)^2(1 + i)$ $= P(1 + i)^3$
Lth year .	—	—	$P(1 + i)^L$

It will be seen that the amount at the end of L years,

$$A = P(1 + i)^L \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and hence the total interest earned,

$$I = P\{(1 + i)^L - 1\}^* \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the compounding is performed more often than once a year the result will pile up at a greater rate; thus a rate of 8 per cent. per annum compounded quarterly for three years is equivalent to twelve compoundings of 2 per cent. each time and amounts to $1{\cdot}02^{12} = 1{\cdot}268$, almost as much as if it were $8\frac{1}{4}$ per cent. compounded annually, or 9 per cent. simple interest. More generally if i' is the actual rate, compounded n times per annum, the interest period will be $\frac{1}{n}$ years, and the interest for the period $\frac{i'}{n}$. It is then possible to construct a table like the above, but with "first interest period," "second

* The name "compound interest law" and the formulæ given above can be applied to an[illegible] series of this kind, even though having nothing to do with actual [illegible] depreciation is sometimes assumed to follow a compound interest law, and in this case the series is a decreasing one and the plus sign in the formulæ becomes a minus one.

interest period," etc., instead of "first year," "second year," etc. At the end of L years there will have been nL interest periods, and the amount will therefore be

$$A = P\left(1 + \frac{i'}{n}\right)^{nL} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In order to avoid the duplication of formulæ necessary to cover cases in which the interest period is less than a year it is convenient to know the effective rate of interest i which compounded annually will give the same result as the actual rate i' with more frequent compoundings.

$$\text{If } P(1 + i)^L = P\left(1 + \frac{i'}{n}\right)^{nL}, \text{ then } (1 + i) = \left(1 + \frac{i'}{n}\right)^n,$$

$$\text{or} \qquad i = \left(1 + \frac{i'}{n}\right)^n - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus if the actual rate is 8 per cent. compounded quarterly, the effective rate

$$i = \left(1 + \frac{0{\cdot}08}{4}\right)^4 - 1 = 1{\cdot}02^4 - 1 = 0{\cdot}08245, \text{ i.e., } 8\tfrac{1}{4} \text{ per cent.}$$

Moreover, this substitution can be made in any of the other compound interest formulæ, so that in the data for sinking funds, etc. which follow, it will always be assumed that i represents either the actual rate if compounded annually, or the effective rate due to an actual rate i' compounded more than once per year.* It will be seen from example 1 at the end of the chapter that the difference between simple and compound interest in affecting the growth of a sum is very great, but the difference between annual and more frequent compoundings is very slight, and rarely need trouble the engineering student.

In conclusion, it should be noticed that it is the difference between the frequency of compounding and the frequency of distributing the interest to the lender that causes the difference between simple and compound interest. If the interest is paid over the moment it is generated, or with the same frequency that it is reckoned up, no difference would exist. It is for this reason that compound interest is not favoured by the law, since it is the duty of the creditor to demand his interest as soon as it becomes due.

* A further assumption in this and succeeding formulæ is that the compounding takes place an exact number of times per year. This assumption can be taken to cover all the cases likely to be met with in practice.

Present and Future Worth: Discounts.—The present worth of a future sum of money (or service valued in money) is merely the principal which will amount to that sum at the end of the specified time and with the specified rate and kind of interest. Thus if the present worth be denoted by P and the future amount by A the formulæ given above will apply without alteration. The future worth of a present sum is, conversely, the amount (A) which that sum will reach, so that the sum, in this case, forms the principal (P) in the formulæ (see summary on p. 291).

It is sometimes necessary to compare alternatives on the basis of present or future worth, and in such cases all the items involved must be separately valued, their dates of expenditure or income determined, and then each sum must be "projected" on to the present or future date fixed upon. Thus in finding the present cash value of patent rights, or of certain purchases or sales to be made at future dates, the method of present or future worths is an extremely useful one. (See examples 2, 3 and 6.)

True discount is the total interest, *i.e.*, the difference between the present worth of an amount and the amount itself, and is denoted by I in formulæ (1) and (4). The word "discount" is largely employed in connection with bills of exchange, which are promises to pay a stated amount (called the face value) at the end of a stated period, during which time interest is to be reckoned at a certain rate. Thus the holder of a bill for £110 (face value) due in two years' time and at 5 per cent. simple interest who wishes to receive its ready money value will have to accept something less than the full face value. The true discount would be £10, and if this were allowed the holder would receive £100, this being the true present worth which will amount to £110 under the stated conditions.*

Annuities and Sinking Funds.—The cases so far considered have been for single sums of money, and it remains to consider

* In practice the bankers or brokers who discount bills charge more than this for their services, and what is called "bankers' discount" is the interest computed on the *face value* of the bill (*i.e.*, £11 in the above case). Thus the bank gains and the seller loses two years' use of the interest, since the latter pays *in advance* interest which is reckoned on the face (that is to say, future) value. Another way of putting it is to say that the true discount is the present worth of the bankers' discount, *i.e.*, of the interest computed on the face value of a bill.

Bankers' discount is not of particular importance to engineers, but in order to avoid confusion between the two methods of computation the term "total interest" will generally be employed instead of "discount."

cases involving a series of payments. The name "annuity" is given to any series of uniform periodic payments, usually of a constant amount each year, although the actual payment may be made quarterly, half-yearly, etc. The three most usual types are—perpetual annuities, continuing for ever; annuities on a person's life, which terminate at death; and annuities for a fixed number of years (or to realise a fixed amount).

Examples of the annuities which continue in perpetuity may be seen in the pensions occasionally granted by a grateful Government to a man and his heirs for ever; and approximate examples can be seen in the ground rents of leasehold property running for a long period (*e.g.*, 999 years). Although involving an infinite series, they are actually the easiest to calculate, since they are obtained by permanently investing such a sum that the interest each year equals the required annuity. There is here no question of compound interest, since the money is paid out (whether yearly or more often) as the interest materialises. Thus the present worth or investment which will yield a perpetual annuity of D per annum is $\frac{D}{i}$. When a bridge or other structure is made of such materials that it can be assumed to last for ever and to be permanently useful, it is possible to calculate from its first cost and by using this perpetual annuity formula what should be the annual charge made for it. (*Cf.* example 4.)

Examples of the second class of annuity can be found in most life insurance and endowment schemes. These do not come much within the scope of the present work, and, moreover, they can generally be considered as special cases of the third class (fixed period or fixed amount annuities), calculated on the average expectation of life. The third type of annuity is, therefore, the most important and the one chiefly met with in engineering problems, the sinking fund being a particular case of this type.

When it is desired to provide out of annual income for some future event requiring a large single expenditure, it is usual to lay by an equal sum each year so that it may amount to the required quantity in the known period. Thus if land is being used whose lease runs out and requires repurchasing (or machinery which requires replacing) in L years at a cost of £A, it will be usual to provide for this by equal annual deposits into a fund which is then called a sinking fund. These deposits are so calculated that at the end of the period they

amount to £A, which can then be withdrawn and applied to the necessary repurchase, or to extinguish or amortise the liability in question.

If the deposits were merely stored in a strong box until the required time the annual deposit figure would clearly be $\frac{A}{L}$, but if, as is usually the case, they are profitably invested, the actual figure will be less than this, by an amount depending on the rate of interest, the length of time and the frequency of

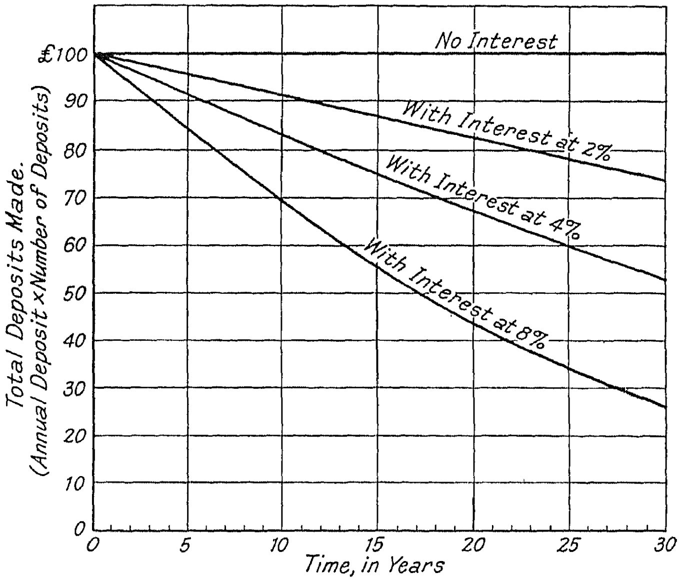


FIG. 1.—Total Paid into Sinking Fund. (To realise £100 in the time shown: interest compounded annually.)

compounding. In Fig. 1 the total deposits made in order to realise £100 (*i.e.*, the annual deposits multiplied by the number of deposits) are plotted against time for a number of rates of interest and will show very clearly the advantage to be gained from a profitable investment of sinking funds. Even with interest at 4 per cent. and compounded only once a year, the total money deposited in a thirty-year fund is only a little over half the sum realised.

Sinking Fund Formulæ : Annual Deposits.—The most important calculation in connection with sinking funds is to find the

future worth of any given set of deposits—that is to say, the total amount to which they will build up at the end of the period. Usually the problem arises in the converse direction. A given amount A is required at the end of a period of L years—what must be the annual deposit D to realise it, with a given method and rate of interest? In order to arrive at these results it will be well to tabulate a simple case, putting down the present and future worth of each payment made. It will be understood that the present worth refers to the worth at the commencement of the first year, and the future worth that at the end of the last (or L^{th}) year.

Deposit D made at the end of each year for L years.

Interest rate i per annum (actual or effective) compounded annually.

Date referred to as	Lapse of Time.	Deposit made.	Present Worth (now) of this Deposit (which would amount to D in		Future Worth (at end of Period) of this Deposit (to which D will amount in	
Now	→-.-.-.-					
	1st year					
	-.-.-.-	D	1 year)	$\frac{D}{1+i}$	L–1 years)	$D(1+i)^{L-1}$
	2nd year					
	-.-.-.-	D	2 years)	$\frac{D}{(1+i)^2}$	L–2 years)	$D(1+i)^{L-2}$
	3rd year					
	-.-.-.-	D	3 years)	$\frac{D}{(1+i)^3}$	L–3 years)	$D(1+i)^{L-3}$
	-.-.-.-	D	L–2 years)	$\frac{D}{(1+i)^{L-2}}$	2 years)	$D(1+i)^2$
	(L–1)th year					
	-.-.-.-	D	L–1 years)	$\frac{D}{(1+i)^{L-1}}$	1 year)	$D(1+i)$
	Lth year					
End of Period	→-.-.-.-	D	L years)	$\frac{D}{(1+i)^L}$	0 year)	D

The items in the last column, reading from bottom to top, m a geometrical progression, each factor being obtained

from the previous factor by multiplying by the ratio $(1 + i)$. The sum of any such series is given by

$$\frac{\text{Last term} \times \text{Ratio} - \text{First term}}{\text{Ratio} - 1}$$

and hence the total amount of the sinking fund

$$A = D\,\frac{(1+i)^L - 1}{i}, \text{ or } D = \frac{Ai}{(1+i)^L - 1} \quad . \quad . \quad (7)$$

If the interest is compounded more than once a year (say, at the rate of i' compounded n times per annum) then each deposit will amount to $D\left(1 + \frac{i'}{n}\right)^{n \times \text{time}}$ and the final formula will be

$$D = A\,\frac{\left(1 + \frac{i'}{n}\right)^n - 1}{\left(1 + \frac{i'}{n}\right)^{nL} - 1}$$

A simpler method, however, is to use the previous formula, letting i represent the effective, annually compounded, rate found from equation (6). In any case the effect of more frequent compounding is slight, and can usually be neglected in sinking fund calculations (see example 7).

In a similar way the items in the first column of the above table form a geometric progression, and the total present worth (P) of the series of payments is given by

$$P = D\,\frac{(1+i)^L - 1}{i\,(1+i)^L} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This formula occurs wherever a terminable concession of any kind, *e.g.*, the exercise of patent rights, the use of a monopoly, the lease of a mine, etc., is purchased for a lump sum in advance (see example 6). It is also necessary in cases where money is borrowed on promise of repayment in instalments spread over a period of years—a common practice in building society mortgages or hire purchase terms.

The previous formula, No. (7), may be regarded as the fundamental equation of a sinking fund, and is of the very greatest importance. It represents the annual end-of-year cost of providing for the renewal or redemption of any piece of apparatus liable to depreciation, and hence it occurs in the great majority of the problems to be dealt with (see example 7 and numerous examples in later chapters). The symbol d will

be used to denote the deposit necessary to realise unity $\left(=\frac{i}{(1+i)^L-1}\right)$, and in Appendix II., p. 293, the percentage values ($100d$) are tabulated for a number of rates of interest and lengths of life. Although in this chapter formulæ have been developed in every case and used in the worked examples in the earlier part of the book, this is only for the sake of completeness and to show the student how the figures are derived. The engineer in practice will, of course, use tables such as the one referred to.

If the deposits are made at the beginning of each year instead of at the end, they will each have a year longer to run, and will each amount to $(1+i)$ times what they would have become. The total amount of the series will therefore be increased in this ratio, or, conversely, the deposit will be correspondingly decreased if the amount is to be the same, giving the formula

$$D = \frac{Ai}{(1+i)^L-1} \times \frac{1}{1+i} \quad . \quad . \quad . \quad . \quad (9)$$

By similar reasoning the present worth of each deposit will be higher in the same ratio, since each individual present worth has one year less in which to grow into the deposit D. This gives the formula

$$P = D\frac{(1+i)^L-1}{i\,(1+i)^{L-1}} \quad . \quad . \quad . \quad . \quad (10)$$

It will be noticed that if the amount A is fixed instead of the deposit D, the present worth will be exactly the same whether payments are made at the beginning or at the end of the year, since the smaller size of the deposits will then balance the shorter time each present worth has to run. This is obvious also from the fact that the present worth of the series of deposits is also the present worth of the final amount A, so that P and A can be calculated from each other directly by the application of formula (3).

Sinking Funds: Non-Annual Deposits.—The formulæ developed above only cover cases in which the deposit is made either at the beginning or end of each year, and in which the interest is either compounded annually, or, with more frequent compounding, has been converted into its equivalent annually-compounded figure. It now remains to consider cases in which the deposits are made at other than yearly intervals, and in

what follows the intervals of time which elapse between successive deposits and those which elapse between successive compoundings of the interest will be referred to as the deposit interval and the interest period respectively.

The first case to be considered, and one which admits of very easy solution, is that in which the deposit interval coincides with the interest period, as for example when both occur quarterly or half-yearly. In this case the formulæ developed above can be employed as they stand, provided that for L is substituted nL, *i.e.*, the number of interest periods instead of the number of years, and for i is substituted $\frac{i'}{n}$, the interest per period instead of the yearly interest. Thus a ten-year sinking fund deposited quarterly and compounded quarterly at 8 per cent. per annum would be treated as if it were a forty-year fund at 2 per cent. per annum, and the necessary end-of-period deposit would be given by $D = \frac{A \times 0{\cdot}02}{(1{\cdot}02)^{40} - 1}$. The proof of this follows directly from the work given above with the substitution here suggested, and a similar formula can be developed for deposits made at the beginning of each period.*

Another special case, which however rarely arises, is that in which the deposit interval is less than the interest period, *e.g.*, if deposits are made at the beginning or end of each month and the interest compounded quarterly. The meaning of this would be that each deposit earns simple interest until the end of the next compounding period, and compound interest thenceforward.† Special formulæ can be developed to meet these cases, but as they do not often arise it is generally preferable to group the payments in such a way that they can be handled by one of the simpler expressions already mentioned. Thus with monthly deposits and quarterly compounding, it is possible to group each set of three payments and find what quarterly amount they are equivalent to, say, at the *end* of each interest period. This equivalent deposit can then be treated as in the case mentioned above (deposit interval = interest period).

For all other cases—in which the deposit interval is some

* The general formulæ are Nos. (11) and (12) below, with $k = n$.

† If *no* interest is earned by the deposits until the compounding period, they can be thought of as one deposit made at the end of each interest period, but split up into portions for convenience in payment.

multiple of the interest period—the following formulæ must be used :—

Let i' be the rate of interest, compounded n times per annum.

Let k be the number of deposits made per annum (n must equal or be a multiple of k).

Then — necessary end-of-interval deposit

$$D = A \frac{\left(1 + \frac{i'}{n}\right)^{\frac{n}{k}} - 1}{\left(1 + \frac{i'}{n}\right)^{nL} - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

And — necessary beginning-of-interval deposit

$$D = A \frac{\left(1 + \frac{i'}{n}\right)^{\frac{n}{k}} - 1}{\left(1 + \frac{i'}{n}\right)^{nL} - 1} \times \frac{1}{\left(1 + \frac{i'}{n}\right)^{\frac{n}{k}}} \quad . \quad . \quad (12)$$

These formulæ cover all cases in which the deposit interval is equal to or greater than the interest period, even if the interval is more than a year (k fractional), but in every case the periods must start or finish at the same point. The proof of these follows exactly the lines of the previous section.

Hire Purchase.—There is an application of the above formulæ which is becoming increasingly important to engineering salesmen. In dealing with articles of domestic application, such as cookers, vacuum cleaners and washers, particularly where these are of high initial price and have to compete with cheaper alternatives, it is important to be able to quote hire purchase terms as an alternative to payment in cash. The usual meaning of hire purchase is that the customer pays down only a fraction of the total cost, and he then receives the article and is allowed full use of it. The payment of the balance is made by means of a number of equal instalments spread over some pre-arranged period, at the end of which time he becomes the full legal owner of the article. It will be evident that the instalments must be large enough to amount to the original cash price plus such extra as may be needed to cover interest, etc.

There are several advantages in this. Even if the percentage added for interest and collection expenses represents a high

figure per annum, it will not amount to very much in the period of a few months or years that the money is outstanding, and the terms can therefore be made attractive to the prospective purchaser who is short of ready money. This also gives the vendor a chance to keep in touch after purchase and advise as to the correct methods of use, and it can often be combined with some sort of guarantee or maintenance throughout the paying period, which is very reassuring to non-technical users. It also enables the relatively high first cost of, say, electric cookers to be offset against the saving in running costs, both of these being spread over a period of time.

Regarded generally, the practice of hire purchase (within reasonable limits) has the effect of setting in motion the wheels of industry at comparatively little risk of inflation or boom, with its resulting rise in prices. For although virtually it consists of the creation of credit or purchasing power, it at the same time stimulates production, and so results in more goods to be purchased. Regarded as a loan, the rate of interest per annum can be high without appearing oppressive, since the time is generally fairly short, and the full amount is not outstanding even for the whole of the time. Moreover, the lender (that is, the salesman, or the bank or insurance company he employs) has adequate security, since he receives a deposit or initial payment before handing the goods over, which is usually not less than 10 per cent. of the cash price. He has also the right to confiscate the article if the customer gets in arrears with his instalments, although usually making some allowance for the payments already made.

With regard to the calculation of hire purchase problems, the interest period can usually be assumed to coincide with the repayment interval, thus simplifying the formulæ which have to be employed. As the whole time is relatively short, and the rate has to cover various additional expenses and possibly some risk, the basic period can conveniently be one month—thus a rate of 1 per cent. per month would represent, over a length of time, 12 per cent. per annum compounded twelve times each year. In such a case if P represents the sum originally outstanding, *i.e.*, cash price less initial deposit, then, applying equation (8),

$$P = D\,\frac{(1+i)^L - 1}{i\,(1+i)^L} \text{ or the monthly deposit } D = \frac{Pi\,(1+i)^L}{(1+i)^L - 1}$$

where i is now the interest per month and L the number of months.

With the above-mentioned rate of interest (1 per cent. per month), if there is £100 to be paid back in twenty-four end-of-monthly instalments, $D = £100 \frac{0{\cdot}01 \times (1{\cdot}01)^{24}}{(1{\cdot}01)^{24} - 1} = £4\ 14s.\ 2d.$ per month. As the purchase is usually spread over some simple even number of payment intervals, it is an easy matter to employ a table which shows at once the necessary figure to charge per £1 of purchase value. The formula will therefore rarely be employed, but, as in previous cases, it has been given so that the student may know the origin of the figures he employs, and may check or vary them at will.

Worked Examples

General Note on Examples.—In all the examples involving sinking funds, etc., occurring in the earlier chapters, formulæ are employed, and these are referred to by number, exactly as they occur in the text. In the later examples the table on p. 293 is used, as this will of course be the normal procedure in practice.

1. How long will it take a sum of money to double itself when accumulating at 4 per cent. interest: (a) If simple interest; (b) if interest is compounded annually; (c) if interest is compounded quarterly; (d) if true compound interest (infinitely short intervals)?

(a) Equation (2) $A = P(1 + Li)$ and $A = 2P$

$$\therefore\ Li = 1 \text{ or } L = \frac{1}{i} = \frac{1}{\cdot 04}, \textit{ i.e., } 25 \text{ years.}$$

(b) Equation (3) $A = P(1 + i)^L$

$$\text{or } L = \frac{\log \dfrac{A}{P}}{\log(1 + i)} = \frac{\log 2}{\log 1{\cdot}04} = 17{\cdot}67, \textit{ i.e., } 17 \text{ years } 8 \text{ months.}$$

(c) Equation (5) $A = P\left(1 + \dfrac{i'}{n}\right)^{nL}$

$$\text{or } nL = \frac{\log \dfrac{A}{P}}{\log\left(1 + \dfrac{i'}{n}\right)} = \frac{\log 2}{\log 1{\cdot}01} = 69{\cdot}66.$$

$$\therefore\ L = 17{\cdot}41, \textit{ i.e., } 17 \text{ years } 5 \text{ months.}$$

Or, by finding the effective rate—equation (6),

$$i = \left(1 + \frac{i'}{n}\right)^n - 1 = 1{\cdot}01^4 - 1 = 0{\cdot}0406,$$

and applying equation (3), $L = \dfrac{\log 2}{\log 1{\cdot}0406} = 17{\cdot}41$ as above.

(d) Page 12: $A = Pe^{Li}$, or $e^{Li} = \dfrac{A}{P} = 2$,

hence $Li = \log_e 2 = 0{\cdot}693$

$$\therefore\ L = 17{\cdot}33, \textit{ i.e., } 17 \text{ years } 4 \text{ months.}$$

2. A quantity of wood is stored for resale in two equal batches of £500 worth each, the first batch after five years and the second after eight years. If interest is at 5 per cent. compounded annually, what is (a) the total present worth of the store; (b) the total future worth at the end of the eighth year?

(a) From equation (3),

$$P = \frac{A}{(1+i)^L} \quad . \ . \ . \text{ first batch} = \frac{500}{(1{\cdot}05)^5} = 391{\cdot}76$$

$$\text{second batch} = \frac{500}{(1{\cdot}05)^8} = 338{\cdot}42$$

Total $730{\cdot}18$, *i.e.*, £730 4*s.*

(b) $A = P(1+i)^L$ . . . first batch $= 500\,(1{\cdot}05)^3 = 578{\cdot}8$

second batch $= 500$

$1078{\cdot}8$,

i.e., £1,078 16*s.*

3. A man is allotted shares to the total purchase price of £1,000 on which he has to pay one-tenth down, and the remainder in two equal instalments in one and a half, and three years respectively. What is the total present worth or liability of the transaction reckoning interest at 6 per cent. compounded half-yearly?

Present worth of immediate payment $= \frac{1}{10}$ of £1,000 = £100

,, ,, £450 in $1\frac{1}{2}$ years (from equation 5),

$$P = \frac{A}{\left(1+\frac{i'}{n}\right)^{nL}} = \frac{450}{\left(1+\frac{\cdot 06}{2}\right)^{2\times 1\frac{1}{2}}} = \frac{450}{(1{\cdot}03)^3} = £411{\cdot}8$$

Present worth of £450 in 3 years =

$$\frac{450}{\left(1+\frac{\cdot 06}{2}\right)^{2\times 3}} = \frac{450}{(1{\cdot}03)^6} = £376{\cdot}9$$

Total £788·7,

i.e., £788 14*s.*

4. A concrete structure presumed to last for ever is estimated to have a permanent usefulness valued at £500 per annum. If the rates and upkeep amount to £200 per annum, what is the maximum which can economically be paid for the structure in the first place, with money at 6 per cent. per annum ?

Net annual sum available to pay for structure = £300.
This is a perpetual annuity (p. 16), whose present worth

$$= \frac{D}{i} = \frac{£300}{0{\cdot}06} = £5{,}000.$$

5. What will be the amount of a sinking fund, accumulating at 5 per cent. interest annually compounded, made by depositing £50 (a) at the end and (b) at the beginning of each year for twenty years ? What will be the difference if the deposits are made in instalments of £25 at the end or beginning of each half year, and if the interest is compounded quarterly ?

First Part

(a) From equation (7)—

$$A = D\frac{(1+i)^L - 1}{i} = £50\frac{(1{\cdot}05)^{20} - 1}{0{\cdot}05} = £1{,}653$$

(b) From equation (9)—

$$A = D\frac{(1+i)^L - 1}{i} \times (1+i) = £1{,}653 \times 1{\cdot}05 = £1{,}736$$

Second Part

(a) From equation (11)—

$$A = D\frac{\left(1+\frac{i'}{n}\right)^{nL} - 1}{\left(1+\frac{i'}{n}\right)^{\frac{n}{k}} - 1} = £25\frac{(1{\cdot}0125)^{80} - 1}{(1{\cdot}0125)^{\frac{4}{2}} - 1} = £1{,}691$$

(b) From equation (12)—

$$A = D\frac{\left(1+\frac{i'}{n}\right)^{nL} - 1}{\left(1+\frac{i'}{n}\right)^{\frac{n}{k}} - 1} \times \left(1+\frac{i'}{n}\right)^{\frac{n}{k}} = £1{,}691 \times (1{\cdot}0125)^2 = £1{,}734$$

6. A patent expiring in seven years' time is estimated to be worth £2,000 a year. What is the present cash value of the patent if interest is reckoned at 6 per cent. compounded annually, assuming that the gain resulting from the use of the patent accrues at the end of each year ?

Equation (8)

$$P = D\,\frac{(1+i)^L - 1}{i\,(1+i)^L} = £2{,}000\,\frac{(1\cdot06)^7 - 1}{0\cdot06 \times (1\cdot06)^7} = £11{,}160.$$

7. A machine costs £1,000, and has a useful life of twenty years, after which its value is assumed to be zero. If interest is at 5 per cent. compounded annually, how much must be put aside at the end of each year in order to replace the machine when its life terminates ? What difference would it make if interest were compounded four times per annum ?

Equation (7)

$$D = \frac{Ai}{(1+i)^L - 1} = \frac{1{,}000 \times 0\cdot05}{(1\cdot05)^{20} - 1} = 30\cdot2, \ \textit{i.e.}, \ £30\ 4s.$$

Equation (6)

$$\text{Equivalent rate } i = \left(1 + \frac{i'}{n}\right)^{n} - 1 = (1\cdot0125)^4 - 1 = 0\cdot051$$

$$\text{Hence} \qquad D = \frac{1{,}000 \times 0\cdot051}{(1\cdot051)^{20} - 1} = 29\cdot98, \ \textit{i.e.}, \ £29\ 19s.\ 6d.$$

8. An electric washing machine is priced at £40 cash, or alternatively it can be bought for a deposit of 10 per cent. and thirty end-of-month instalments. If interest and other expenses are covered by a figure of 1 per cent. per month, what must be the amount of the monthly instalment, and by how much will the total paid exceed the cash price ?

Value of initial loan $= £40 - 10\% \doteq £36$.

From Equation (8) $D = \dfrac{Pi\,(1+i)^L}{(1+i)^L - 1}$, where i is now the interest per month $= 0\cdot01$, and L is now the number of months.

$$= \frac{36 \times 0\cdot01\,(1\cdot01)^{30}}{(1\cdot01)^{30} - 1} = \frac{0\cdot36 \times 1\cdot348}{0\cdot348} = 1\cdot395, \ \textit{i.e.}, \ £1\ 7s.\ 11d.$$

Total paid over the whole period will be 30 × £1 7*s*. 11*d*. + £4 = £45 17*s*. 6*d*., *i.e.*, £5 17*s*. 6*d*. more than the cash price if paid at the commencement.

CHAPTER II

TOTAL DEPRECIATION

Divisions of Subject.—The subject of depreciation can conveniently be considered under four heads :—

Concerning Total Depreciation.	*Physical and Economic Aspect.*—This concerns what depreciation is, what it depends upon, and hence what is the total provision which must, economically, be made to meet it. *Financial Aspect.*—Discussing what forms this provision may take.
Annual or Interim Depreciation.	*Accounting Aspect.*—Concerning the method of allotting this total liability amongst the years concerned. *Legal Aspect.*—Income tax and valuations.

The first two aspects concern only the total depreciation which takes place and have nothing to do with the distribution of the depreciation over the years of life : they are dealt with in the present chapter. The other two concern the course of depreciation during the life, and are dealt with in the following chapter.

Causes.—At the beginning of the book capital was defined as the working equipment or store of wealth used in producing more wealth. Some items of this store, such as freehold land, assets held as security, etc., can be regarded as permanent—that is to say, they are not altered either in quality or extent by use or the passage of time, and, although market fluctuations affect their value, such changes do not enter into technical "depreciation." The other items of the capital store, usually comprising much the greater part, are characterised by a steady persistent diminution in value, and are termed by accountants "wasting assets."

The causes which bring about this loss in value are very various, one of the chief in the case of tangible assets such as plant, machinery, buildings, etc., being the wear and tear due to their use. With this should be grouped the diminution of

a natural resource, such as a mine whose total yield is limited, or an estate which is being gradually sold in lots. The characteristic of the loss in value in all these cases is that it is roughly proportional to the amount of use—*i.e.*, to the output of the equipment. In engineering there are many types of plant and equipment which have to be renewed with a frequency depending chiefly upon the amount of their use, *e.g.*, machine tools, locomotives and rails, batteries, lamps, etc.

The second group of causes is that due to natural disintegration and decay, with which might be put the risk of accident, theft, earthquake, etc., in fact, all those natural processes or contingencies which may happen to goods stored where moth and rust corrupt and thieves break through and steal. The principal examples in mechanical and electrical engineering are the corrosion and ageing of metals, and the deterioration of insulators ; and it will be noticed that the common feature in this group is that they are all roughly proportional to time. It will be seen, of course, that this division is a simplification of the actual facts, since practically no items are a function of time alone or of use alone. The deterioration of an insulator, though mainly a matter of time, depends also upon temperature and therefore use, whilst items normally proportional to use are also affected to some extent by time.]

Obsolescence and Inadequacy.—A third group of causes of depreciation is that due to changes in manufacturing processes or public habits, tastes, etc., or in the circumstances of the user, which may put the equipment " out of date " before either of the other causes has made it worn out. This group is less easy to estimate than either of the others, since although partly proportional to time, it is also dependent upon the progress of invention and engineering knowledge and technique. Any new undertaking involving the borrowing of money and its conversion into manufacturing equipment is dependent for its profitableness upon a nice calculation as to the relative suitability of various machines and the probable public demand for the product. Even though these are all correct at the time of investment they may be upset by a new development or discovery, and it may then be worth while to " scrap " machinery that is still in working order.

It will be seen that this type of depreciation is distinguished from those previously considered by the fact that it is a change not in the asset itself but in its environment, which destroys its

economic suitability. It just as certainly results in the asset becoming "left behind," but it is due not to the asset itself getting worse but to the alternatives getting better. It is sometimes called "functional depreciation" to distinguish it from the two types of physical depreciation already considered, and it in turn can be divided into two groups, obsolescence and inadequacy. Thus a small factory owner starting with a 10 h.p. gas engine may before the end of its life find it obsolete, and more economically replaceable by a 10 h.p. electric motor. On the other hand, starting with a 10 h.p. unit he may find it necessary to expand, and advisable to purchase a 20 h.p. size. In this latter case the smaller machine is not obsolete so far as external judgments go, but it is inadequate and can with economy be superseded. But it is doubtful whether the division into two groups is really necessary, since both require the same treatment and economically amount to practically the same thing. Moreover, both represent a species of out-of-dateness, *i.e.*, an asset once suitable is now (through no fault of its own) uneconomic as compared with the available alternatives. In what follows the word "obsolescence"* will therefore be used in a broad sense to cover both these groups of non-physical depreciation.

Some authorities hold that changes of this sort, which are entirely external and involve no alteration in the assets themselves, cannot strictly be regarded as depreciation. This is in itself a somewhat academic objection, but there is also a case on practical grounds for excluding obsolescence risks from the depreciation provision, which may be summarised briefly as follows :—

Suppose that the life of a machine, apart from external improvements, is twenty-five years, and that the depreciation reserve is based upon this estimate. If the progress of invention is such as to make the machine obsolete after twenty years, although still in working order, presumably the replacement will not be made unless it pays to do so ; and in this case the value of the other five years of life can be debited to the earlier years of the new machine, *i.e.*, paid for by the saving which it effects.

In the author's opinion this is a mistaken argument, as

* The phrase "functional depreciation" seems particularly unfortunate. An engineering asset is nothing unless it functions, so that physical life and physical depreciation can refer only to this. The peculiarity of obsolescence is that the asset, though still capable of functioning, is no longer economic in that form and place, and is therefore superseded.

presumably the improvements, if general, will in the long run benefit the consumer, *i.e.*, they will lower prices and so take away the possibility of recouping the cost through the higher efficiency. To put it in another way, the original undertaking should be in a position to earn the same profits on its capital after twenty years as a new concern starting at the later time and purchasing the more up-to-date plant. Hence the original enterprise should protect itself against probable developments by estimating its plant life as twenty years—only if it is proof against the average expectation of future inventions can it be said to be justified in embarking upon manufacture.

Moreover, the distinction between the changes in the asset itself which are included in ordinary depreciation, and the external changes which produce obsolescence, is not really a helpful one, and it is better to regard the tendency to become out-of-date as just as much an attribute of the asset as the tendency towards decay in other respects. Certain materials are liable to wear, rust or deteriorate, and certain classes of plant are liable to become superseded ; and it is as much the business of a new undertaking to provide against the one as against the other. The vital thing about an asset is its usefulness, which depends upon factors both intrinsic and extrinsic.

To sum up, it may be said that from the purely practical point of view—that of providing against all usual depreciation risks, there seems no reason for excluding obsolescence, providing it arises as a normal consequence of the use or existence of the asset, or is a reasonably anticipated risk of the enterprise. In a rapidly moving industry such as electrical engineering, some degree of obsolescence is to be expected and must be allowed for. Another and very practical objection to excluding this item from the depreciation allowance is a purely financial one, namely, the difficulty of finding the necessary capital to pay for the earlier replacements. When the plant life has been estimated as twenty-five years and provision made accordingly, if it is then found to be economically sound to replace at twenty years (the extra profits more than paying for the loss of five years' life), it still may not be possible to do so, since the depreciation fund will not have matured to the necessary value, and fresh capital may not be easily obtainable.

Another suggestion which has been made is that there should be two separate and distinct funds, one a renewals fund to provide for replacement at the end of the "physical" life, and the other an improvements fund to provide for the shortage

in the event of its owner deciding to replace it earlier, *i.e.*, at the end of its "economic" life.* One objection to this is that it is difficult enough to get most engineers to establish one satisfactory fund for each item of his plant, and to get him to establish two such funds (even if in practice it merely means more book-keeping) is asking too much. But even on purely theoretical grounds it is doubtful whether there is a case for two separate funds.

The theory on which this suggestion is based is that the physical life is something definite and inherent in the item itself, whereas the economic life is an external concept which it is in the power of the owner to change. But on examination, even the physical life † is seen to be no such definite finite period, but is almost as much in the owner's power to determine as the other. Oliver Wendell Holmes wrote of a "wonderful 'one-hoss shay'":— ‡

> "That was built in such a logical way
> It ran a hundred years to a day."

At the end of which time it went absolutely to pieces :—

> "All at once, and nothing first—
> Just as bubbles do when they burst."

No doubt this is the aim of all design, but it is an aim seldom realised. Some types of plant certainly have a major portion which, when it goes, does so completely—thus it may be said that the life of a steam locomotive is that of its boiler, and the life of a battery is that of its plates, but even here the case is not quite simple, since a battery may be renewable by replacing half the plates. With most other plant the case is even less simple, as there are many different parts, some small and renewed under maintenance, some large and rarely needing renewal, but when renewed giving the plant a fresh lease of life.

In fact there are many types of engineering plant which can be patched up so as to go on functioning almost indefinitely. There is then no definite physical life, and the decision as to

* See F. Gill and W. W. Cook, "Principles Involved in Computing the Depreciation of Plant," *Journal I.E.E.*, vol. 55, February, 1917.

† The word itself is misleading in this connection. Organic beings have a definite life, and the moment that spark is extinct the usefulness in that particular form is ended. Machinery, unless it blows up or collapses altogether, has usually no such definite end to its existence.

‡ "The Deacon's Masterpiece," in "The Autocrat of the Breakfast Table," by Oliver Wendell Holmes.

when to scrap (on whatever grounds) is entirely an economic one and within the owner's power to determine. This is particularly so with electrical apparatus, since the actual passage of current or flux produces little or no permanent deterioration or distortion. To take an extreme case, a wireless receiving set built of metal, mica and ebonite (apart from fittings) might easily continue to function for fifty or a hundred years. Yet there is nothing whose real depreciation in value is certain and so rapid, and no one would suggest a fund dated for fifty years or more, by which time the whole of wireless technique and practice may be revolutionised. To say that such a set has a physical life of a hundred years and an economic life of five or ten years is to introduce unnecessary confusion. For an engineer the life of particular plant can only mean one thing, namely, its useful life in that form taking into consideration all probable factors; and to provide against this, one single fund should and usually must suffice.

In conclusion, it is necessary to distinguish between the non-physical depreciation or obsolescence referred to above, and the depreciation or otherwise of non-physical assets. In discussing the various causes affecting depreciation, the assets chiefly visualised have been the tangible physical objects such as machinery, ships, mines, etc. But as will be seen from later sections, there are many other kinds of asset making up the capital store, and some of these, such as goodwill, are also essentially temporary and can be regarded as subject to persistent diminution in very similar ways to those outlined above. This is more particularly true in the case of firms dealing with luxuries and novelties, in which a new undertaking often sinks considerable sums in advertising and establishing trading connections. Unless maintained this goodwill is liable to perish, and thus one finds a certain make of boot polish which was a household word in Dickens' time disappearing entirely. In the case of engineers dealing with necessities or public services, the reverse process may occur—instead of capital being spent in making a "splash," whose effect will gradually peter out, the goodwill is built up by services and so increases with time. It will be noted that in some contracts (tramway terms, p. 69) all consideration of these non-physical assets is expressly excluded. This is because the goodwill consists in this case not so much of public estimation as of monopolistic rights, which being granted for a definite period have come to an end at the time of purchase.

Definition and Characteristics.—Having reviewed the main causes which bring about what is known as depreciation, it will now be possible to state more precisely the particular meaning which the word bears here. In ordinary language, depreciation may refer to any reduction in value, but in its technical sense and as used in this book it refers only to "expired capital outlay" or "the fall in value of wasting assets arising out of their use or tenure." * It therefore relates only to the first cost, the time and action of the enterprise, and to any normal and anticipated external changes. If the original capital outlay is denoted by C and the final salvage value by V, then the total depreciation is $C - V$. (The quantity $C - V$ is sometimes spoken of as the original or total "wearing value" of the asset.)

Several important points follow from this definition of total depreciation, one of which is that depreciation has nothing to do with market fluctuations after the asset is purchased (except in so far as these affect salvage values). The change in value covered by the definition can only be in one direction, namely, a loss, and the common expression by which shares or property is said to appreciate or depreciate with changes in the market is the general and not the technical use of the word. Since depreciation covers only the "eating away" of capital assets as the express result of the purposes for which they are held, it follows that any arbitrary fluctuations due to outside changes must be regarded as "profit" or "loss," whereas depreciation is a necessary working expense which must be met before any profit can be said to exist.

Another thing which follows from the above definition of depreciation is that it has nothing directly to do with the cost of replacing the asset, although the depreciation provision will normally be used for that purpose. If the plant originally installed cost £1,000 at the commencement of its life and is worth £100 at the end, then the requirements of strict depreciation have been fully satisfied if £900 are forthcoming at the end of the period; and if it is then desired to replace it by plant costing £1,500, the extra should be financed by the additional profits which the better plant should earn. On the other hand, if the extra cost is likely to be necessitated for identical plant owing to general market changes during the life of the first plant, it is legitimate and only right to increase the

* See especially "Depreciation and Wasting Assets," by P. D. Leake.

strict depreciation provision by a reserve fund (financed by the extra profits which a market rise should produce) to meet this anticipated extra cost.

Summing up, it will be seen that the question of whether depreciation provision should be varied to suit probable cost of replacement cannot be settled by any hard and fast rule, but must be a matter of individual judgment. If the rise in prices is general, such as that due to a war, and takes place not too late in the life of the original plant, an increase in the provision seems reasonable, since the plant has virtually become more valuable and profit-earning. But if not, it means charging present customers for future burdens.

It will be noticed that this is another aspect of the question already discussed under obsolescence—namely, whether the depreciation provision should concern itself with changes *external* to the asset itself. It is a question to be settled on practical rather than theoretical grounds, and depends largely on whether or not the change is general or normal to the enterprise under the particular circumstances ruling. Thus when installing plant utilising new processes or technique, it is to be anticipated that later inventions will still further improve things, making necessary either a shortened economic life (earlier "scrapping") or a larger outlay on replacement. This loss in value of the asset in the form of accelerated or accentuated obsolescence may be said to arise out of, or be conditioned by, the nature and processes of the enterprise itself, and therefore can legitimately be considered as coming strictly within the definition of depreciation.

A final point which follows from the above definition is that it is only *total* depreciation which can be said to have any precise meaning and determinable value. The normal processes of an enterprise imply the purchase of wasting assets for the express and sole purpose of their gradual destruction in the making of profits. The assets are regarded as out of the market because permanently locked up in that process until the termination of their useful life, and the question of what they are worth or would fetch in the meantime does not enter, since such a sale is no part of the normal activities of the enterprise.

Unfortunately it is not possible to defer all consideration of the financial position of an enterprise until its wasting assets have been completely exhausted; and it is therefore necessary to allocate in some manner the total capital which will expire,

amongst the years of useful life which the plant is expected to have. This point is discussed in the next chapter, but it is important to notice here that any such apportionment of total depreciation is necessarily arbitrary and artificial. It is impossible to say how much has gone in any one year, since there is usually no rigid and immediate connection whatever between loss of value and passage of time. Plant may be, and often is, every bit as useful and (apparently) as valuable any one year as it was the previous year—all that can be said is that it is one year nearer to its end.

Life and Salvage Value.—When depreciation has proceeded to such a point that the machine or equipment in question is no longer economically serviceable in its original capacity, it may be said to have served its useful or economic life (L). It has been seen that this may be due to its being worn out, dangerous, or obsolete, but in any case the owner is presumed to be well informed and without prejudice, so that he removes it as soon as it becomes uneconomical as compared with the available alternatives. Its value immediately before removal is called its salvage or remainder value (V), and would be estimated in the ordinary way by what a purchaser would pay for it as it stood.

It will be noticed that this is lower (by the amount of the removal costs) than the value after removal, or the value of the constituent parts if unusable as a whole. (It is well to keep the term "scrap value" for these latter values, although too often the words "scrap," "salvage," "residual" and "remainder" are used without distinction.) It follows from this that the salvage value may be zero or even negative—representing a liability rather than an asset—as in the case of a chimney or other tall structure which costs more to remove than its parts are worth, yet which has to be removed owing to being a danger to the public.

It will be seen from the above that, using the symbols already defined, the total depreciation in a life of L years can be expressed as $C - V$, and the correct depreciation provision is that which amounts to this sum at the end of L years. Provided this sum is available at the end of the life, and provided the interest on the original capital has been maintained throughout these L years, then the enterprise is solvent and has cleared off all its liabilities in connection with the asset in question. For adding the value of the salvage V the total sum C is then

in hand, and can be used either to pay off the original loan, leaving no outstanding debt, or else to purchase a new asset of the same sort as before.

In the second case a new period of L years is initiated, the original debt remains but is balanced by a fresh asset, and the annual cost is the interest on this debt plus the contributions to the new depreciation fund which must be instituted for the new asset. In calculating the cost of the service rendered by any asset, it therefore does not matter whether it is assumed that the asset is employed for its one life-time and the undertaking wound up, or whether it is assumed that the service is continued indefinitely, provided that in the latter case a new asset replaces the old one every L years.

In the above discussion the life of the asset has been spoken of as though it were a fixed determinable quantity, whereas in fact the life of each particular asset, like the life of a human being, is an individual affair determined by the particular circumstances. Moreover, as the provision for meeting depreciation has to be initiated at the commencement of the life, it follows that all depreciation funds on account of tangible assets are based upon an estimate, namely, the anticipated useful life of the asset in question. Depreciation funds on account of terminable concessions such as ground leases, patent rights, licences, etc., are simpler in this respect, since in such cases the exact life period is known from the beginning.

It was seen in discussing causes of depreciation that whilst many of these are roughly proportional to time, some are dependent upon the use which the equipment gets. Thus the steel rails of a tramway undertaking may have to be renewed every eight years with heavy traffic, or may last twice as long if the traffic is light, and in a similar way the life of a mine or other natural store of exhaustible material will depend upon the amount extracted each year. In spite of these items, time is so much the most convenient basis for economic calculations that it is almost invariably used in estimating depreciation; * but of course the time must be assessed having in view the probable amount of wear, and may have to be adjusted as the undertaking proceeds. On p. 294 are tabled the estimated lives of various plant by different authorities, and although these

* The chief exceptions are those in which th[illegible] are so small that no interest is charged, and hence there need be [illegible]mo-rate charges. Thus, in the case of lamps (Chapter X.), life is reckoned in burning hours, and this is really a service unit rather than a time unit.

may have to be modified to suit individual differences of wear and climate, it is surprising how little such differences emerge in comparison with the steady inevitable decline due to general causes.

Another point which is often surprising to the user is that there is no essential connection between the performance of a machine and either its life or its salvage value. Even the most perfectly operating machine will sooner or later have to be superseded, and though its efficiency and working order may be as good as on the day it was installed, it will then be worth little or nothing owing to the difficulty of finding a purchaser whom it will exactly suit. With electrical machines in particular, owing to the wide diversity of possible variations (size, type, voltage, frequency, etc.), the poor salvage value of plant apparently almost " as good as new " is an item which must be reckoned with.

The fundamental fact which emerges from the physical and economic aspect of a business employing wasting assets, is that an adequate part of the total capital expiry must be assumed to have taken place each year of the useful life of the asset; and this must be made good or paid for as an essential manufacturing or overhead expense before any review of the financial position becomes possible. This necessity applies to every variety of wasting asset purchased for capital sums, not only plant and machinery (on their estimated lives), but leases, patents and other concessions * (on their known periods), and even goodwill, which is in constant danger of depreciation from rival firms.

Thus if a man owns a business in which £5,000 is invested in tangible assets, and which he has worked up until it brings in £2,000 a year, and if he then sells it to a company for £15,000, the latter will be paying £10,000 for the goodwill. But the goodwill, unless maintained by extensive advertising, will probably not last longer than ten to thirty years, or at the outside fifty years; and quite apart from the provision made for the tangible equipment, it also should be written off within a reasonable period, unless it is being fully and continuously maintained. Exactly the same would apply if the plant had been purchased for £5,000 and the goodwill had been built up later by means of a £10,000 advertising campaign.

* *E.g.*, a monopoly of electricity supply for a definite period, if purchased for a lump sum.

Depreciation Provision: Sinking Fund Method.—The particular financial arrangement which is made to meet depreciation is not a matter of great moment to the engineer, since all such arrangements (provided they are adequate) must amount to the same thing, economically, in the end. The primary requisite in any such arrangement is that it shall realise the required sum $(C - V)$ at the end of the estimated life of L years. The next point of importance is that it shall distribute its burden as equitably as possible between the several years of the plant's useful life—preferably to correspond either to its annual earning capacity, or else to the estimated amount of its annual wear or capital expiry. (The latter is chiefly a matter of accounting; see on p. 63.)

There are cases in which no profits are expected to arise during the earlier years. A good example is in connection with the electricity supply or telephone service in an undeveloped area, for which plant is installed to meet anticipated future needs, and in excess of present requirements. The same case also arises in a rubber or other estate requiring development. Conversely there are cases in which the service rendered by the plant will materially lessen with time, this being a frequent occurrence in connection with apparatus such as storage batteries, road vehicles and other traction equipment, in which decreasing performance or increasing interruptions are features of the latter part of the life. Under special circumstances such as these it may be convenient to have a depreciation provision which is unequally loaded and which interposes a heavier burden on either the later or the earlier years. In the vast majority of cases, however, the advantages obtainable from the use of an equipment will be found to be tolerably equal for each year of its useful life, and hence it is generally convenient for depreciation provision to take the form of equal annual payments.

From what has been said it will be clear that a sinking fund built up during the useful period by equal annual payments suitably invested forms a very convenient type of depreciation provision. The life of such a fund should coincide with the anticipated useful life of the equipment, and substituting this figure for L in any of the formulæ on pp. 19 to 22, and putting the amount required $(C - V)$ equal to A, it is a simple matter to calculate the deposit necessary for any particular rate of interest and deposit interval. This method of provision has some degree of official sanction in connection with financial

proposals for electricity stations, and it is also extensively employed by municipal authorities as a method of extinguishing their loans, such extinction (in the case of the trading branches) being to a large extent in the nature of a depreciation provision. Even when this method is not actually employed, the sinking fund forms the best basis for calculation in estimating true costs. (See examples 1, 3 and 4.)

Some explanation is necessary as to the actual mechanism of a sinking fund. In the calculations in the later part of this book the expense of depreciation is always computed on a sinking fund basis, *i.e.*, it is assumed that for each and every machine or portion of a machine whose choice is a matter of individual selection, a sinking fund is initiated at the commencement of its life such as to yield $(C - V)$ £ at the end. Whilst this assumption is necessary for economic calculations, it will be obvious that the process is not carried out literally for each separate item. All that is necessary for sound finance and accounting is for the depreciation account to be credited each year with the appropriate deposits for every item of plant according to its cost and estimated life. From the same account would be taken, as required, the sums necessary for replacement of the plant for whose depreciation the fund exists. The outstanding balance each year * between the incomings and outgoings of this account (which would not necessarily be large in the case of a diverse and long-standing enterprise) would then be invested as profitably as possible, consistent with safety and ease of withdrawal.

In the case of small or privately owned items the assumptions made as to the initiation of an individual sinking fund will seem even further from the facts of the case, and may appear to make many of the calculations which follow somewhat academic. Thus the purchaser of a motor car will probably *not* establish a sinking fund to extinguish its cost, nor will he necessarily borrow the money for its purchase. But in using the money to buy a car it means that he cannot use it elsewhere (where it could certainly earn interest), and, in fact, he very possibly withdraws it for the purpose from some easily accessible, interest-bearing deposit. And if he repays this withdrawal out of income during the five years of use, those

* The phrase "each year" is used to indicate any convenient period—in large enterprises it is the usual accounting practice to make credits to the reserve every month.

repayments will accumulate interest and be equivalent, economically, to a sinking fund.

The important point in all cases is that the assumptions made shall represent not necessarily the actual financial procedure, but the essential economics of the case. It is then possible to calculate what a given service is really costing each year, and thus to compare it with the available alternatives. Only if the engineer can calculate exactly what any particular machine will cost per annum can he base his decisions and make his choice on sound lines, and only so can he be said to be a complete engineer.

Composite Life.—In the case of an enterprise employing a few large assets, the parts of which do not require separate renewal (*e.g.*, a coal mine), it is no doubt preferable to have one fund maintained entirely separate from all other finance and kept intact until the end of the period. When one such fund has to serve for several items of plant having different lengths of life, it is necessary to find the "composite" or "equated" life of the group of items, *i.e.*, that life which when applied to the total cost would give the correct total provision. Strictly, there is no such thing as a composite life for depreciation purposes, since it varies slightly with the type of provision made. It is, however, possible to make a sort of average of the lives of the different assets, based on their relative costs and frequency of renewal, which would be exactly right if depreciation were computed on the basis of laying by annual deposits without interest (the "straight line" or "strong box" basis), and which will be approximately correct for the sinking fund basis.

The composite life can be found as follows :—Multiply the wearing value ($C - V$) of each asset by its relative frequency of renewal to give the "cost product," and then multiply this product by the life so as to give a second quantity which might be called the "life-cost" (or simply "life") product. The sum of all the life products divided by the sum of all the cost products will then give the composite life. This method is illustrated in the first six columns of the table below in finding the composite life of two assets, A having a life of twenty years and wearing value £2,000, and B having a life of ten years and a wearing value of £1,000.

In order to illustrate the correctness of this method of estimation the last two columns show the annual deposits

which would be necessary on the straight line and on the sinking fund basis respectively, if a separate fund were established for each asset. These add up to £200 and £140 respectively, whilst the single deposits for an asset of £3,000 having the composite life of fifteen years would be £200 and £139 respectively. Whilst it is always legitimate to calculate the necessary deposit on the basis of a single asset having the composite life, the objection to having only a single fund

Asset.	Wearing Value ($C - V$).	Relative Frequency of Renewal.	Cost Product.	Life.	Life Product.	Annual Deposits Necessary.	
						Strt. Line Method.	Sinking Fund Method (at 5 per cent. interest).
	£					£	£
A	2,000	1	2,000	20	40,000	100	60·5
B	1,000	2	2,000	10	20,000	100	79·5
		Totals	4,000		60,000	£200	£140

$$\text{Composite life} = \frac{60{,}000}{4{,}000} = 15 \text{ years.}$$

locked up (*i.e.*, invested externally) for this period is that the time when the actual replacements are required may not coincide with the maturing of the fund. It is therefore preferable to have a running depreciation account in the manner described in the previous section. If the correct credit is made to this each year for every item of plant reckoned separately, there will be no need to calculate the composite life, and there will always be enough in the fund to renew each item as it falls due.

Depreciation Provision: Loan Reduction Method.—When the original loan or some definite portion of it was incurred in order to purchase some particular machine or equipment, it seems a very logical procedure to pay off that loan in stages as the machine or equipment wears out or otherwise loses value. The undertaking is then left at the end of the economic life, not with the original loan balanced by a sinking fund of approximately the same amount, but with no loan at all. (See example 2.)

It will be seen that the differences between the two methods are more financial than economic. The method outlined above involves making arrangements for periodic repayment of portions of the loan, and in order to avoid discrimination between the different shareholders it is not unusual for the repayment to be made by drawing lots.* Such a repayment may sometimes embarrass the shareholders themselves, but it has some good financial effects, *e.g.*, in steadying the market for that particular share and preventing violent fluctuations in price. Also, against the inconvenience of having to pay off some shareholders every year, must be put the corresponding inconvenience, with a sinking fund, of having to re-invest some money each year.

The loan repayment method avoids the difficulties which occur with sinking funds in connection with the security and interest upon the sums which are being put by. Security and interest are inter-related, and in general terms it may be said that it is difficult to find investments for a sinking fund which, whilst enabling regular amounts to be taken up each year, will combine absolute security with a rate of interest as high as that paid on the original loan. Thus a concern may have to raise its initial loan at a comparatively high rate owing to novelty or supposed insecurity, and may have to accept a much lower rate on its sinking fund investment. This difficulty is not so serious in the case of a large long-standing enterprise employing plant of various lengths of life, as here it has been seen the capital items tend to cancel out, *i.e.*, the investments which are necessary can be made "in the business," in the purchase of fresh plant requirements. For such an enterprise the sinking fund method would appear to be preferable, the loan reduction system being only suitable when the assets approximate to one length of life.

There is a legal restriction which should also be mentioned when discussing the pros and cons of the loan reduction method. In practice this method is virtually restricted to the repayment of debentures, since the existing English law does not recognise the right of any joint stock company to reduce its share capital without going through various formalities each time the repayment is proposed and then obtaining the sanction of the court. Debentures are the fixed interest bearing

* A not uncommon announcement in a prospectus is that certain shares are to be paid off at par by equal annual drawings of such and such an amount. Another plan is to purchase shares in the open market (especially if and when the price is low) and then extinguish them.

shares which are presumed to cover all the tangible and realisable assets of a company, and whilst it is primarily these assets whose depreciation has to be provided for, this legal restriction interposes considerable limits to any great extension of this method of provision.

Summing up, it may be said that the loan reduction method, when it can be adopted, has distinct advantages over the sinking fund, since it is equivalent to a rate of interest equal to that on the original load, combined with absolute security. It is, however, generally less convenient and frequently impracticable.

Loan Reduction : Instalment or "Straight Line" System.—With regard to the rate at which repayment is made in the loan reduction method of depreciation provision, there are two plans which can be adopted. The more usual one is for the loan itself to be paid off in equal amounts or instalments, interest being at the same time paid on the value of the loan still outstanding. This has the disadvantage that it interposes a heavier burden during the earlier years, as at this time the undertaking has to subscribe not only the constant repayment sum, but also the interest on almost the full amount of the loan. It has the advantage of simplicity and ease of explanation to shareholders, since a fixed amount of scrip will have to be cancelled each year. It also makes for satisfactory accounting, since, if depreciation is assumed to follow a straight line law during the intervening years, the amount of loan outstanding will at all times be equal to the assumed value of the plant.

In order to compare economically the sinking fund and loan reduction methods, it will be necessary to take into account the total yearly capital expenses of the asset, namely, the cost of hiring (or interest) as well as the cost of replacing (or depreciation). For simplicity it will be assumed that the salvage value is zero, so that the amount A in the formulæ on p. 19 can be equated to the first cost C. (Unless the salvage value is either zero or some definite proportion of the first cost, it is impossible to group the interest and sinking fund payments together as a function of C.) It will also be assumed that payments are made at the *end* of the year in each case.

The yearly capital expenses under the sinking fund method then consist of an interest charge Ci plus a depreciation charge D, where $D = \frac{Ci}{(1+i)^L - 1}$. Under the "straight line" loan

reduction method the capital expenses will consist of a fixed instalment of loan repayment M, where $M = \frac{C}{L}$, plus the interest on the remaining loan. The latter will be Ci the first year, $(C - M)i = Ci\left(1 - \frac{1}{L}\right)$ the second year, $Ci\left(1 - \frac{2}{L}\right)$ the third, and $\frac{Ci}{L}$ at the end of the last or L^{th} year. (The last mentioned is, of course, the interest for one year on the final outstanding instalment $\frac{C}{L}$.) Thus whilst the total (end-of-year) charge in the case of the sinking fund is a constant figure, $Ci\left[1 + \frac{1}{(1 + i)^L - 1}\right]$, that for the loan reduction by instalments method is $\frac{C}{L}(1 + iL)$ in the first year, $\frac{C}{L}(1 + iL - i)$ in the second, and decreases by equal annual amounts of $\frac{Ci}{L}$ ("straight line" law) to $\frac{C}{L}(1 + i)$ in the last year.

The above formulæ assume that the sinking funds bear interest at the same rate as that paid on the original loan, and on this assumption Fig. 2 has been drawn to illustrate the comparison between these two methods of depreciation provision. In the upper portion of the figure is shown the total yearly cost or capital charge in the two cases, for plant having a first cost of £1,000, salvage value zero, and a life of twenty years, interest being at 5 per cent. throughout. With the sinking fund method (curve S) the charge is the same each year, namely, £80 5*s*., whereas with the loan reduction method (curve R) it falls uniformly from £100 at the end of the first year to £52 10*s*. at the end of the last. The total money actually paid is of course slightly less by this latter method, since greater amounts are paid in the earlier years (hence the two lines do not cross at the centre). But the total present worth of the capital charges is the same in the two cases.

Below the curves R and S, will be seen plotted the accumulated reserve which results in the two cases, *i.e.*, the total amount of loan repaid (R') or the total sinking fund accumula-

tion (S'). It will be noted that the sinking fund builds up at a greater rate during the later years, so that, whilst it is always less than the "straight line" value, it amounts to the necessary £1,000 at the end of the period.

The term "straight line" loan reduction has been employed to

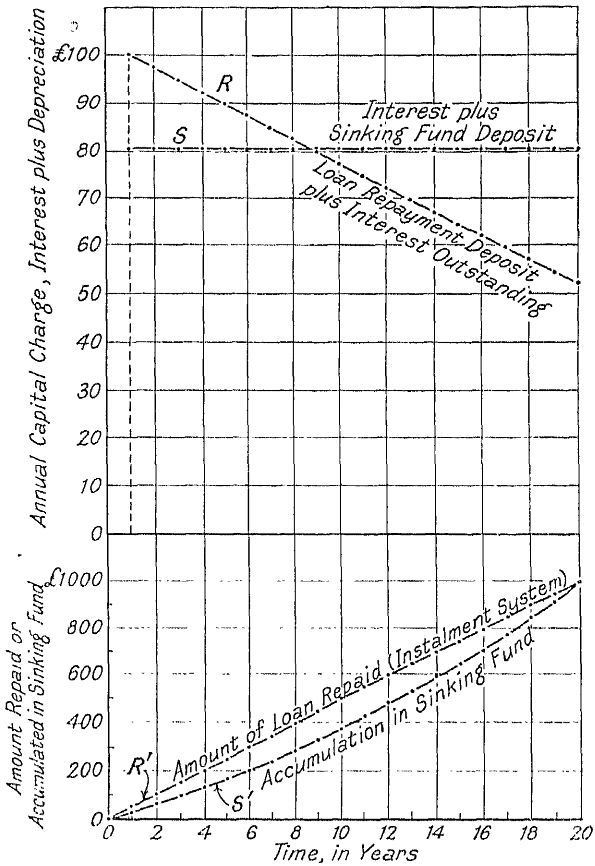


Fig. 2.—Comparison of Methods for Depreciation Provision.
[The dots represent actual payments or amounts.]

describe the principle of this method of depreciation provision, but it need not necessarily involve the actual repayment of loan, any more than the sinking fund method necessarily involves any considerable sunken fund. As already explained, with a large undertaking employing many assets of

varying dates and lives, the depreciation reserve may be merely a current account constantly receiving and paying out, so that what is here called loan reduction chiefly consists of the "writing down" of the plant's value, *i.e.*, the crediting to this fund of the appropriate amount. The essential difference between the loan reduction by equal instalments or "straight line" method and the sinking fund method, is that the former writes off an equal amount each year, but (owing to interest) loads the undertaking more heavily in the earlier years, whereas the latter writes off smaller amounts in the earlier years, but loads the undertaking equally each year.

A very common practice is to credit the depreciation reserve with equal amounts each month or year, but to neglect all interests on these deposits. As in fact interest is always earned wherever the money is, this means that the interest appropriate to these deposits is really credited to the general revenue. This is therefore an example of the "straight line" loan reduction method described above, and results in a heavier burden during the earlier years of the life of the asset, with a correspondingly better showing later on. But by not crediting the interest where it is due, this fact is obscured, and many engineers who employ the "straight line" method imagine, because equal annual instalments are made, that the capital charge for interest and depreciation is the same each year.

Loan Reduction : Annuity System.—Both the advantages of the equal instalment method (in giving a "straight line" loan reduction) and the disadvantages (in heavier burdens during early years) can be avoided by paying back smaller amounts during the earlier years, and gradually increasing the amount to balance the decreasing interest on the outstanding loan.* This is sometimes called the annuity method of loan reduction to distinguish it from the equal instalment method, the word "annuity" (as indicating a series of equal annual payments) referring in this case to the sum of the contributions to repayment plus interest. Even without the proof given below, it should be fairly evident that (assuming constant interest rates throughout) this is identical economically with the sinking fund method—they are merely alternative financial schemes for arriving at exactly the same point.

In order to obtain an arithmetical statement of the annuity

* An approximation to this which preserves the simplicity of the instalment system is to pay off the original loan in equal instalments, commencing after the undertaking has been in operation for several years.

loan reduction method (for end-of-year payments) it will be necessary to start at the far end, and call the amount of loan outstanding at the beginning of the last year V_1, and that two years from the end V_2, etc. The total capital charge is made up of repayment and interest in such proportions as to amount to a constant total d each year :—

COMBINED LOAN REPAYMENT AND INTEREST.

(*First cost* C, *salvage value zero, composite deposit* d *at end of each year for* L *years.*)

Loan outstanding at beginning of :—	Composite deposit d consists of :— Interest on this for one year.	Plus loan repayment.	Hence :—
Last year (*i.e.*, 1 year from end) . . . V_1	$i\,V_1$	V_1	$d = iV_1 + V_1 = V_1(1+i)$ or $V_1 = \frac{d}{(1+i)}$
Last but one year (*i.e.*, 2 years from end) . . . V_2	$i\,V_2$	$V_2 - V_1$	$d = V_2(1+i) - \frac{d}{(1+i)}$ $\therefore\ V_2 = \frac{d}{(1+i)} + \frac{d}{(1+i)^2}$
Last but two years (*i.e.*, 3 years from end) . V_3	$i\,V_3$	$V_3 - V_2$	$d = V_3(1+i) - \left\{\frac{d}{(1+i)} + \frac{d}{(1+i)^2}\right\}$ $\therefore\ V_3 = \frac{d}{(1+i)} + \frac{d}{(1+i)^2} + \frac{d}{(1+i)^3}$
First year (*i.e.*, L years from end) . . . . V_L	$i\,V_L$	$V_L - V_{L-1}$	$\therefore\ V_L = \frac{d}{(1+i)} + \frac{d}{(1+i)^2} + \frac{d}{(1+i)^3} \cdot \cdot + \frac{d}{(1+i)^L}$

The amount outstanding at the beginning of the first year V_L is, of course, the amount of the original loan or first cost C, and in the above table it is seen that this equals

$$\frac{d}{1+i} + \frac{d}{(1+i)^2} + \frac{d}{(1+i)^3} \cdot \cdot \cdot + \frac{d}{(1+i)^L}$$

This particular arithmetical series has already been referred to (p. 19), and its sum

$$V_L = C = d\,\frac{(1+i)^L - 1}{i(1+i)^L}, \text{ whence } d = \frac{Ci\,(1+i)^L}{(1+i)^L - 1}.$$

Now with the sinking fund method, the total annual payment $= Ci + D$, and $D = \frac{Ci}{(1+i)^L - 1}$, so that the total becomes $Ci\left\{1 + \frac{1}{(1+i)^L - 1}\right\} = \frac{Ci\,(1+i)^L}{(1+i)^L - 1}$, which is the same

as d above. Hence the two amount to the same thing from the £ s. d. point of view, and the sinking fund curves in Fig. 2 above represent also the loan reduction by annuity method.

Comparison of Methods.—It has been pointed out already that the depreciation provision is frequently more a question of accounting than of the actual depositing of money. The necessary amount each year for each item of plant is credited to the depreciation fund or reserve, but it is not necessarily invested in a separate fund or used to pay off loans, being frequently employed in the business in place of fresh borrowing, or for the purchase of replace plant as and when this becomes necessary. Hence in the questions at the end of the chapter involving sinking funds, it is usually assumed that such funds earn interest at the rate paid on the original loan, although if these funds have to be separately invested each year, they will usually receive less than this.

It has been seen that there are two main assumptions which can be made for determining the magnitude of these annual sums. The first is that the sums earn interest in a separate fund and are sufficient to amount to the required wearing value at the end of the anticipated life. The capital charges for interest and depreciation are then the same each year, and this rule applies to the sinking fund and also to the loan reduction by annuity methods. The second assumption is that the loan is paid off by equal annual instalments, and this gives a total capital charge which is heaviest at first and diminishes uniformly as the life proceeds.

A third assumption which can be made and which is often convenient for accounting, even when not carried out in fact, is that a fraction of the original loan is repaid or written off each year, the amount being a fixed percentage of the value *at that particular year*. This means that the actual quantity written off gets less each year, and as the interest on the outstanding loan also gets steadily less, this constitutes a method of loading the earlier years even more heavily than by the equal instalments method. Either of these two latter methods may be useful when dealing with plant subject to excessive depreciation in the earlier years, as in the examples already given of storage batteries and road vehicles. They are also useful where market valuations have to be considered, in view of the possibility of the plant changing hands before the end of its economic life.

These two latter methods may be called the equal instalment or "straight line" plan, and the reducing balance or "compound interest" plan respectively, and are useful as assumptions in estimating the interim depreciation for accounting, income tax or other purposes, even when they are not employed as an actual depreciation provision. In this capacity they are treated in the chapter which follows. The last-mentioned method introduces difficulties if it is used to calculate the total capital costs, since both the interest and the depreciation deposits progressively diminish.

Repairs and Renewals.—Some undertakings make no definite depreciation provision, but claim instead that all necessary repairs and renewals are made good out of income as and when they occur. The fallacy of this plan is that it is extremely difficult to "even out" the replacements so that they represent a fairly uniform burden each financial year. Moreover, there is a great danger of the more long-standing items being overlooked, thus giving a deceptive appearance of prosperity during the intermediate years, and necessitating new capital when the items finally become obsolete.

This plan is therefore not to be recommended except when only short-lived items are being dealt with (*e.g.*, lamp renewals would always be treated in this way), and it is chiefly to be found among the smaller concerns or when a number of undertakings are under one management, as with municipal enterprises. Thus a tramway system may find that its track requires renewing every ten years, its rolling stock every fifteen, and so on, so that by taking the routes in turn, an average yearly expenditure can be made, and the whole system maintained at a constant level of value and efficiency.

In the case of municipal enterprises the whole position is enormously complicated by the fact that even with those likely to have a permanent earning capacity, endeavour is made to repay the whole of the original loan within a relatively short period, so that the enterprise may become an interest-free public property at the earliest possible moment. Laudable though such an ambition may be, it is natural that unless the undertaking is exceptionally prosperous there will not be much margin for depreciation provision in addition to repayment. The consequence is that, whilst the debt is extinguished within the time required, it will usually be found that in the meantime depreciation has proceeded so far as to necessitate a fresh loan.

Such a method over-reaches itself, and it would appear a sounder plan to sanction longer period loans coupled with an insistence on a separate and adequate depreciation provision. By this means the finances of each separate undertaking would really represent its correct position and profitableness, and when at the end of fifty years or so the original loan was finally extinguished, the public would know that they owned a concern in as good a state of efficiency as it was at first, or with a sinking fund sufficient to take advantage of any fresh inventions or developments then obtainable.

Even when there is a genuine wish to provide fully against depreciation, it is not always easy to decide what items should be covered by the provision—thus, while the revenue account is each year charged with maintenance under the two heads of "repairs" and "depreciation deposit," the former is intended to be an immediate expenditure on small items, whilst the latter is presumed to be invested so as to be avilable for complete replacement at some distant period. In practice, however, there are many partial replacements which are more than repairs and which, if paid for out of hand, will unduly load the year in which they occur, whilst at the same time extending the life period of the complete plant beyond the original estimate.*

It is clearly impossible to draw a rigid line, and to some extent it will be a question of policy, some firms preferring to keep every portion of the plant fully up to the mark and as far as possible up-to-date, whilst others rely more on a short-dated depreciation provision with a view to complete replacement. As a rule, all items which recur every few years or oftener, and which are not large enough to embarrass the year's accounts unduly, can conveniently be regarded as current repairs and paid for out of hand; whereas additions, betterments, and renewals of all major items of the plant whose choice is a matter of individual selection, should be paid for out of the capital account (depreciation reserve), a new fund being then instituted for the new part. It therefore depends in part upon the magnitude as well as the frequency of the renewal; thus in a storage battery, new acid or alkali, even though required only once or twice in a lifetime, would usually

* The author has a fountain pen, which has had at various times two new caps, a new nib, and a new feed. If at any time the barrel breaks and is replaced, there will be none of the original pen left, yet it would be difficult to say at what point the pen's life ceased. Certainly at this rate it need never be scrapped, and there are many old engineering equipments which are almost in a like case.

be regarded as maintenance, whereas new plates would be virtually a new battery and would be purchased out of the depreciation fund established for the purpose.

Naturally the repairs and renewals policy must be taken into consideration when estimating the total useful life and salvage value of each item, since a generous policy of replacements out of revenue may keep the plant in such a condition that the depreciation fund practically only has to provide against obsolescence. In any case, however, the estimates should assume that the plant is maintained in *working* order out of revenue.

A final note should be made regarding structures having an indefinitely long life, such as concrete buildings in a country not subject to earthquakes. It is very usual to establish no depreciation reserve for such an asset, but it must not be forgotten that even a structure which lasts for ever may not be eternally useful, since the needs and the technique of future generations are difficult to forecast. The structure known as the "Government Building" in the Wembley Exhibition of 1924–25 had every appearance of being likely to last for ever, but its useful life for the purpose for which it was erected was two years. A sounder plan would therefore appear to be to make *some* provision for every asset, only excepting freehold land, which is not likely to depreciate unless the population declines.

Capital Charges : Economy of Replacement.—Before leaving the subject of total depreciation and proceeding to the accounting and legal problems connected with interim depreciation, it will be well to summarise briefly the economic position so far as this has been taken. With the data given above and in the previous chapter, it is now possible to state what exactly is the cost of possessing a particular capital asset. For convenience in payment, this cost is spread over the useful life of the asset, and this is convenient also for calculation purposes, in which the *annual* cost is the usual basis. Unfortunately actual lives vary, and the payment is therefore spread over the probable useful life, estimated beforehand. The total cost of the asset can therefore be expressed as L end-of-year payments, each totalling Ci units for hire,* plus $(C - V)\dfrac{i}{(1 + i)^L - 1}$ for

* It will be noted that there is here no question of compound interest, since it [illegible] that the accounts are made up and the interest is paid over [illegible] or more often) whenever it becomes due.

replacement; and using the word "charge" to indicate any annual payment, the above can be referred to as the total capital charge.

In order to visualise the problem it will be well to assume an actual case—say the one graphed on p. 47, in which the asset is a machine of first cost—£1,000, salvage value zero, and estimated life twenty years. Assume also that the payments are equally spread over the twenty years (*e.g.*, by sinking fund or loan repayment annuity method) and are made at the end of each year, with interest at 5 per cent. throughout. The capital charge for such an asset is then £80 5*s*. and has to be paid each year for twenty years. It will be noted that the fact of an estimated life of twenty years means not that the asset will necessarily last this length of time and then collapse, but simply that the proprietors of the enterprise have decided to pay for it by twenty equal annual instalments. Whatever time the asset actually lasts, the payments will last the twenty years and can then cease. If the asset is scrapped after eighteen years, there will be two more contributions to pay (or one composite amount), the outstanding balance owing at that moment being £148 (Fig. 2). If used for longer than twenty years there will be no capital charges for any of the additional years—for if interest is still being paid on the original loan on account of first cost, there will be a sinking fund or other investment of equal value on which interest is being received.

If at the end of its life the machine breaks down completely or is otherwise put out of commission, it will necessarily have to be replaced; and if this happens before the twenty-year period the annual accounts will have to be debited, as explained at the beginning of the next chapter. If, on the other hand, the machine outlasts its allotted span and continues to function but with decreasing effectiveness, the question arises as to what is the annual cost of replacing it by a new machine, and what must be the latter's superiority in order to justify its installation at any particular moment. The original machine, whether fully paid off or not, does not enter into the problem, as in any case the arrangements for paying it off have been made and must be proceeded with—the only question is, what is the cost of installing a new machine now, instead of, say, in five years' time, assuming that the old one can continue to function so long.

For simplicity assume that the new machine also costs

£1,000 and lasts twenty years; then its cost will be £80 5*s*. per annum and will last for twenty annual contributions. Moreover, the actual life, whatever it turns out to be, will start from its date of installation, whether that takes place now or in five years' time. So that every extra year that the old machine can hold out (whether it has passed its allotted span or not), it is saving £80 5*s*. per annum, or 8·02 per cent. of the first cost.

Hence the principle may be enunciated that the saving in not replacing one machine by a second is the interest and depreciation on the second machine's capital cost, and this saving takes place each year that the replacement is delayed. In other words, the capital charge for replacing one machine by a second, whenever it is done, is the interest and depreciation on the second machine. For this to be economical, the loss on the first machine during the year it is replaced and all subsequent years, must exceed the annual loss on the second machine (averaged over the whole of its anticipated life) by sufficient to pay for the above-mentioned capital charge.

The last point requires some further elucidation. It might be found that after a certain number of years, machine (1) was costing over £80 5*s*. a year in inefficiency and repairs which machine (2) would save, but it does not follow that the replacement should then be made. For machine two may in its turn show a diminishing efficiency, so that the immediate advantage of replacement may be partly the inherent advantage of the earlier years, which will in any case be reaped whenever the change is made. For the change to be made earlier rather than later it is necessary for the annual saving by the second machine averaged over the *whole* of its estimated useful life to exceed the annual interest and depreciation on its capital cost. (See example 5.)

In considering such a replacement there are, therefore, two points requiring attention. Firstly, it is necessary to calculate when the change should economically be made, and, secondly, if the decision is to change before the completion of the depreciation fund on the first machine, this fund must be made up or maintained until completed, in addition to the new fund instituted for the second machine. These two points are quite unconnected, and must be considered independently.

Worked Examples

1. What is the annual expense or capital charge of owning a machine whose first cost is £400, salvage value £20, and life eighteen years, if money is worth 6 per cent. compounded annually ? What difference will it make if money is compounded four times per annum ?

Interest charge $= Ci = £400 \times 0{\cdot}06 =$ £24 0*s*.

Depreciation charge [End-of-year deposit, equation (7).]—

$$D = \frac{(C - V)i}{(1 + i)^L - 1} = \frac{380 \times 0{\cdot}06}{(1{\cdot}06)^{18} - 1} = 12{\cdot}3, \textit{ i.e.}, £12\ 6s.$$

Total . . . £36 6*s*.

If money is compounded four times a year, the equivalent interest in the sinking fund formula becomes (equation 6)

$$i = \left(1 + \frac{i'}{4}\right)^4 - 1 = (1{\cdot}015)^4 - 1 = 0{\cdot}0638, \textit{ i.e.}, 6{\cdot}38 \text{ per cent.},$$

and this can be used in equation 7 as above.

Or using equation (11),

$$D = (C - V)\frac{\left(1 + \frac{i'}{n}\right)^{\frac{n}{k}} - 1}{\left(1 + \frac{i'}{n}\right)^{nL} - 1} = 380\ \frac{(1{\cdot}015)^4 - 1}{(1{\cdot}015)^{4 \times 18} - 1} = 11{\cdot}86,$$

i.e., £11 17*s*., instead of £12 6*s*.

It will be noted that the change in compounding affects only the depreciation deposit. This is because it is presumed that the profits of the concern accrue regularly throughout the year, so that the payments of interest on first cost can be made quarterly just as easily as annually. Another point to notice is that the salvage value also affects only the depreciation deposit, since whatever the salvage may be worth, it occurs only at the *end* of the life, and interest has to be paid throughout on the full amount of the initial loan.

2. A company is floated with £40,000 $4\frac{1}{2}$ per cent. debenture stock which is to be paid off in forty years by equal amounts at the end of each year. Calculate the total capital charge entailed by the debentures during the first, the twentieth and the fortieth year respectively, and compare with what it would have been had the debentures been redeemed instead by a forty-year sinking fund at the same rate of interest, annually compounded.

Annual instalment of loan repayment $= \frac{£40,000}{40} = £1,000.$

Interest charge 1st year—

$$= £40,000 \times 0{\cdot}045 = £1,800.$$

Interest charge 20th year—

$$= £40,000 \times 0{\cdot}045 \left(1 - \frac{19}{L}\right) = £1,800 \left(1 - \frac{19}{40}\right) = £945.$$

Interest charge 40th year—

$$= £1,800 \left(1 - \frac{39}{40}\right) = £45.$$

Hence total charge 1st year = £2,800.
 ,, ,, ,, 20th ,, = £1,945.
 ,, ,, ,, 40th ,, = £1,045.

By sinking fund. Annual end-of-year deposit for forty-year fund—

$$D = \frac{Ai}{(1+i)^L - 1} = \frac{40,000 \times 0{\cdot}045}{(1{\cdot}045)^{40} - 1} = \frac{1,800}{4{\cdot}816} = 373{\cdot}73, \textit{ i.e., } £373\ 15s.$$

Annual interest payment (constant throughout) = £40,000 × 0·045 = £1,800.

Hence total capital charge by sinking fund method = £2173 15*s*. every year.

3. A machine costs £600, and is estimated to last fifteen years. An alternative machine costs £1,000, and is estimated to last twenty years, and to save £30 a year in maintenance as compared with the former machine. Find which will prove the cheaper installation, taking interest at 5 per cent. and assuming machines have no salvage value.

		£ per annum.
First Machine—Interest	$= 600 \times 0{\cdot}05$	= 30
Depreciation deposit	$= \frac{600 \times 0{\cdot}05}{(1{\cdot}05)^{15} - 1} = \frac{30}{1{\cdot}079}$	= 27·8
Extra maintenance		30
	Total .	87·8

i.e., £87 16*s*. per annum.

Second Machine—Interest	$= 1,000 \times 0{\cdot}05$	= 50
Depreciation deposit	$= \frac{50}{(1{\cdot}05)^{20} - 1}$	= 30·24
	Total . .	80·24

i.e., £80 5*s*. per annum,

which is less than the first machine by £7 11*s*. per annum.

4. Two machines costing £1,000 and £1,300 respectively are each capable of performing a given task, but the former costs 3*d*. per working hour more in labour. If each machine lasts twenty years and has a salvage value of 5 per cent. of its first cost, find how many hours per week must be worked for the two alternatives to balance? Take interest at 6 per cent. on the original loan and at 4 per cent. compounded quarterly on the sinking fund, and assume all other items to be the same in the two cases.

Extra interest on dearer machine—

$$= 300 \times 0{\cdot}06 = \overset{\text{£ per annum.}}{18}$$

Extra wearing value of dearer machine—

$$(0{\cdot}95 \times 1{,}300) - (0{\cdot}95 \times 1{,}000) = £285$$

To realise this in twenty years requires—

$$285 \frac{(1{\cdot}01)^4 - 1}{(1{\cdot}01)^{80} - 1} = 9{\cdot}5$$

Total 27·5

i.e., £27 10*s*. per annum.

This represents 27·5 × 80 threepences, *i.e.*, 2,200 hours, which is 44 hours per week for 50 weeks per annum.

5. During 1910 an equipment costs £100 in upkeep and losses, and this increases by £10 a year in subsequent years. A new equipment to do the same work would cost £1,000 and last twenty-five years, with a final salvage value of £50, and the yearly cost of its upkeep and losses (averaged over its whole life) would be only £50. When will it pay to instal it, assuming that capital is worth 6 per cent. annually compounded?

$$\text{Interest charge on new equipment} = 1{,}000 \times 0{\cdot}06 = \overset{\text{£ per annum.}}{60}$$

Depreciation charge on new equipment

$$= (1{,}000 - 50) \frac{0{\cdot}06}{(1{\cdot}06)^{25} - 1} = 17{\cdot}3$$

Upkeep = 50

Total . . . 127·3

i.e., £127 6*s*. per annum.

Old equipment costs—during 1910, £100
,, 1911, £110
,, 1912, £120
,, 1913, £130

Hence it will pay to replace it at the end of 1912.

6. A motor car (A) costs £225, its annual tax is £12, and its running cost is $2d.$ per mile. A second car (B) costs £110, has a tax of £20, and its running cost is $2\frac{1}{4}d.$ a mile.

It is estimated that after five years' use car A will be worth one-third of its purchase price, and car B will be worth one-fifth of its purchase price. Assuming that the intending owner wishes to retain the car for five years, and that in all other respects the two are identical, find for what annual mileage the two will cost the same.

[Take interest at 6 per cent. on first cost, and neglect interest on depreciation deposits.]

Car A—

		£ per annum.
Annual Interest	$225 \times 0{\cdot}06$	$= 13{\cdot}5$
,, Depreciation, neglecting interest on deposits . . (*i.e.*, "straight line" basis.)	$\dfrac{2/3 \times 225}{5}$	$= 30{\cdot}0$
,, Tax		$= 12{\cdot}0$
Total . .		$55{\cdot}5$

Car B—

Annual Interest	$110 \times 0{\cdot}06$	$= 6{\cdot}6$
,, Depreciation	$\dfrac{4/5 \times 110}{5}$	$= 17{\cdot}6$
,, Tax		$= 20$
Total . .		$44{\cdot}2$

Annual differences in overhead costs	$11{\cdot}3$

i.e., £11 6*s.* per annum.

To cover this at the rate of $\frac{1}{4}d.$ per mile difference will require $11{\cdot}3 \times 240 \times 4 = 1{,}0850$ miles per annum.

CHAPTER III

ANNUAL DEPRECIATION

Interim Valuation: Accounting Problems.—The previous chapter dealt entirely with total depreciation, and it has been seen how this may be provided against by various schemes of contributions during the life of the asset, realising the total expired outlay $C - V$ at the end of this life. It now remains to consider annual or interim depreciation, *i.e.*, the question of how much of the total value ultimately expiring can be considered to have expired during any one year. It must again be pointed out that this question is not one which the engineering economist should normally have to face, since the only provision which it is necessary for him to make is (by definition) that which is necessary to realise the *total* value expiring in the whole life through the normal processes of the enterprise. The two chief reasons for which interim valuations are necessary are, firstly, when a business is sold during the life of an asset, and, secondly, for the annual accounting.

As regards the former, if the value of the assets at the time of sale, as estimated by what the buyer will pay, is less than the first cost minus the sum then accumulated in the depreciation reserve, this does not prove that the latter was inadequate for the purpose for which it was instituted, but merely that it was inadequate for the unanticipated purpose of an interim sale. The difference must then be written off as so much capital loss resulting from the forced sale of assets before the normal end of life. On the other hand, if the interim sale was anticipated from the first, as in the case of many terminable monopolist concessions, such as electricity supply, then the depreciation provision *was* inadequate, as in such a case it should have been the duty of such an enterprise to ascertain at the commencement the method by which the plant would be valued, and to make its depreciation provision at all times square with this.

As regards accounting, it will be realised that amongst the various accounts kept by any undertaking, the income and expenditure is summarised in a profit and loss or revenue

account, and the financial position at the end of each year is summarised in a balance sheet. In the former are shown all receipts for sale of the product, and against these are shown the expenses, including the contribution to the depreciation fund. The balance sheet is debited with (*i.e.*, it shows a liability on account of) the amount of the original loan, and credited with the value of the various assets purchased by it. At the commencement these two will be identical, but as the value of the assets slowly expires, this sheet will also have to be credited with the value of the sinking fund or other investment which has been made to cover depreciation.

In the table below is given a simplified outline (showing only the items referring to tangible assets) of the affairs of an undertaking presumed to commence at the beginning of 1900 by the purchase of plant costing £10,000 and having an estimated life of twenty years and salvage value zero. The revenue account and balance sheet are shown for the first and for the tenth year of working, and it will be found from the data previously given that an annual end-of-year sinking fund deposit of £302 will amount with interest at 5 per cent. to £3,800 at the end of ten years, and £10,000 at the end of twenty years. In the case given, the depreciation is assumed, for simplicity, to follow the sinking fund law (see p. 65), but whatever formula is adopted, the value of the assets must be "written down" by this amount each year if the books are to represent the true condition of the plant and the correct financial position of the undertaking.

Whilst on the question of accounting, some mention should be made of what happens when mistakes have occurred in estimating the total depreciation, through inaccurate forecasts of either the life or the salvage value. If, when it becomes necessary or advisable to replace plant, it is found that the amount of the accumulated fund has not or has more than wiped out the expired value ($C - V$), the difference should be debited or credited to the revenue account for that year. If there is a serious deficit this can be met by a lump sum from the capital account, provided arrangement is made to extinguish this sum by a series of revenue payments spread over the next few years.* (These payments will, of course, be additional to the sinking fund initiated for the new plant.) In no case should

* A simple plan is merely to continue the original depreciation deposits until the full amount has been extinguished, although usually it will be preferable to wipe out the loss straight away.

REVENUE ACCOUNT.

Dr. Losses.	£	Gains.	£
For the year ending 31st December, 1900.			
To Materials and wages .	3,198	By sales .	4,000
„ Interest on debentures	500		
„ Sinking fund deposit .	302		
	4,000		
For the year ending 31st December, 1910.			
To Materials and wages .	3,698	By sales .	4,500
„ Interest on debentures	500		
„ Sinking fund deposit .	302		
	4,500		

BALANCE SHEET.

Cr. Liabilities.	£	Assets.		£
31st December, 1900.				
Registered capital :—£10,000, 5 per cent. debenture stock .	10,000	Plant as purchased in December, 1899	£10,000	
		Less depreciation for 12 months* .	302	9,698
		Amount of sinking fund investment .		302
				10,000
31st December, 1910.				
Registered capital :—£10,000, 5 per cent. debenture stock .	10,000	Plant as valued on 31st December, 1909*	£6,502	
		Less depreciation for 12 months* .	302	6,200
		Amount of sinking fund investment .		3,800
				10,000

* Estimated on "sinking fund" basis.

replacement of plant by items of equal value result in a permanent increase in the outstanding loan.

The above remarks refer only to failures in wiping out past capital expiry and not to additional expenditure in larger and better plant for the future. As already explained, the latter constitutes a definite expansion which may legitimately be financed by additional borrowing (coupled with the institution of a larger sinking fund), since the loan will be balanced by increased assets.

Annual Depreciation Formulæ.—The formulæ given below concern the shape which the depreciation curve follows in its fall from its known value C at the beginning, to its known value V at the end of the useful life of L years, and this question is one on which different authorities are very strongly divided. If the depreciation at any intervening point is defined as the loss in "value" up to the time in question, such a definition does little more than emphasise the different meanings which may attach to the word "value." For example, a perfectly new motor car or cycle taken from stock and run for 200 miles will have suffered in that short time a very big depreciation of exchange value, since it can no longer be sold as new, but as second-hand. On the other hand, to a person who knows its exact history, that it has been properly oiled, etc. (that is to say, to its owner), it is a new machine which has been "run in" to good working order, and is actually more valuable than it was at first. Even if value is interpreted strictly in its economic sense of exchange value (*i.e.*, selling price), there still remains the difficulty in many cases of the absence of any definite second-hand market.

There are three assumptions which can be made in this connection, each of which will result in a fairly simple method of estimating annual depreciation (see Fig. 3 below). Moreover, whilst the first of these, the "straight line" formula, indicates an equal fall in value each year, the second one (the "reducing balance" formula) shows a greater fall in the earlier years, thus corresponding roughly with probable exchange value or selling price, whilst the third, or "sinking fund" formula, falls less rapidly at first, and may often be considered as representing, in the earlier years, the usefulness to the owner.

The "straight line" formula, as its name implies, assumes that the depreciation each year is the same amount—an agreed-

upon proportion of the first cost—so that the value falls in a straight line from its figure of C at the beginning of its economic life to the figure of V at the end. The amount of the depreciation in any one year is then $\frac{C - V}{L}$, and the *rate* of depreciation is $\frac{100}{L}$ per cent. per annum, the rate in this case being expressed per cent. of the wearing value $C - V$.

The reducing balance, or compound interest formula as it is sometimes called, differs from the above in assuming that each year the value falls by a constant percentage of the value *at the beginning of that year*. Thus if this rate were 10 per cent., a structure which cost £100 would be valued at £90 at the end of the first year, £81 at the end of the second, £72 18*s*. at the end of the third, and so on. More generally, if p is the rate per cent. and C the first cost, the value after L years is $C\left(1 - \frac{p}{100}\right)^L$, which is equal to V when L is the economic life. From this it follows that $p = 100\left(1 - \sqrt[L]{\frac{V}{C}}\right)$, which is a bigger rate than that required under the "straight line" law for the same length of life, since it applies to a steadily decreasing principal.

The method of growth or diminution in a quantity in which the rate of change is at all times proportional to the value of the quantity itself is frequently called the law of organic growth or the compound interest law. Hence the second name for this method of estimating depreciation, the word "interest" referring here to the type of law followed and having nothing to do with actual interest on capital. In fact, the sinking fund formula is the only one into which the question of interest, as such, enters.

The compound interest formulæ already given can, however, be used (p. 13). Equation (3) becomes $V = C(1 - i)^L$, since the rate i is negative, and represents the rate on unity and not the rate per cent. indicated by p in the formulæ above. There is, therefore, no need to remember the specialised formulæ just given, and in the questions at the end of the chapter the ordinary compound interest formulæ will be used with a negative sign for the rate.

It will be clear that if the value did actually dwindle in this

way by a definite proportion of its remaining worth, it would only sink to zero in an infinite number of years, the calculation for L being indeterminate unless V has some finite value. Except for the case in which the plant comes to a sudden end and refuses to function after a certain time, this will be a nearer approximation to the facts at this time than is implied in the assumptions of the "straight line" formula. For with the latter, if the plant is not replaced at the end of its economic life its value very quickly becomes zero and then negative, whereas actually so long as the plant is usable it must still have *some* value, even though it would be more economical to remove it. To use again the illustration of a motor car or cycle, after many years' service the owner may find that its frequent breakdowns make it not worth its running costs, so that it is more economical to sell it to a scrap merchant or to some experimenting youngster. But if he is sentimental enough to go on using it, its value at the end of some additional years can hardly be less than nothing, and, in fact, it will probably show a steadily dwindling utility during all the later years of its life rather than the uniform fall and abrupt termination suggested by the "straight line" law. (See examples 1 and 2.)

The third assumption which can be made is that the value of the plant is at all times equal to the first cost minus the amount of the sinking fund accumulated for the purpose of replacing it at the end of its economic life. This particular method of provision has already been fully considered, and if some other method is actually employed, a sinking fund of the right amount must be imagined for the purpose of this calculation. As a sinking fund accumulated by equal annual deposits generates interest proportional to the amount in the fund, it grows at a steadily increasing rate, so that a valuation based on this falls less rapidly at first and more quickly later.

It has been seen that such an assumption may represent the "use value" in the early life of a machine such as a motor car or of plant liable to break up or develop a rapidly decreasing efficiency towards the end of its life. In other cases both utility and exchange value are more likely to be represented by one of the other formulæ. The great advantage of the sinking fund assumption is that when depreciation is being provided for by equal annual deposits (*e.g.*, by sinking fund or "annuity" loan repayment methods) it enables a satisfactory balance sheet to be drawn up, not once per lifetime of the plant, but annually, since the sinking fund assets (or other equivalent

investments) will at all times balance the depreciation liabilities. It will be clear from the table on p. 62 that had annual depreciation been assessed on the "straight line" formula there would have been an adverse balance shown during all the earlier years, unless depreciation provision was made at a heavier rate during these years by means of equal instalments of loan repayment or some equivalent method. From the accounting point of view, therefore, the assumptions made each year as to interim depreciation should correspond to the actual provision

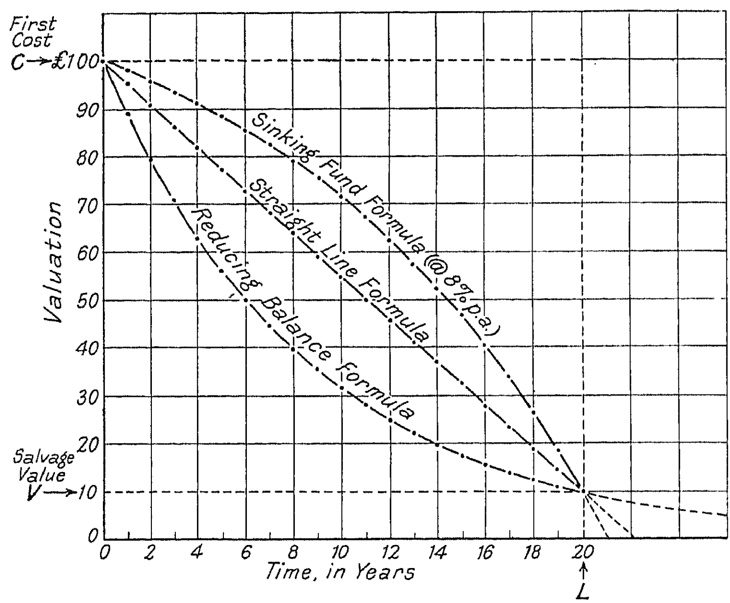


FIG. 3.—Alternative Depreciation Formulæ. [Actual points are indicated by dots.]

established to meet total depreciation, as discussed in the last chapter.

In Fig. 3 these alternative formulæ for estimating interim depreciation are plotted for an asset costing £100 and having a salvage value of £10 at the end of a twenty-year life. In order to exaggerate the curvature of the line representing the sinking fund formula, this latter has been calculated for a high rate of interest, namely, 8 per cent. (annually compounded), but it will be understood that this curve will vary from the straight line to its present position as the interest varies from zero to

8 per cent. In the case of the "reducing balance" or "compound interest" formula, the percentage which it is necessary to deduct each year in order to reach £10 after twenty years is given by $\sqrt[20]{\frac{10}{100}} = 10{\cdot}9$ per cent., so that the amount deducted is £10 18*s*. the first year, £9 14*s*. the second, and so on.

The behaviour of the curves if continued beyond the twenty-year point is indicated by dotted lines, and it will be seen that the value rapidly falls to zero except on the "reducing balance" assumption.

Sale of Undertakings.—In estimating interim depreciation for accounting purposes it is comparatively immaterial which formula is adopted, but when a business is changing hands the matter becomes of very great importance. As an example of the magnitude of the issues which may be at stake, the case may be cited of the National Telephone Company whose licence expired on December 31st, 1911, the agreement being that the business should be transferred to the Government on this date at a price fixed by the then value of the physical assets (*i.e.*, excluding any goodwill: see "Tramway Terms"). Apart from disputes as to the length of life, etc., attributable to each class of plant, the company claimed for depreciation to be computed on a sinking fund basis, whilst H. M. Postmaster-General contended that depreciation should be measured by the ratio which the expired life bore to the whole life, *i.e.*, on a "straight line" basis. Allowing for a half-yearly fund at 5 per cent. interest, the difference between the two methods for the plant in question amounted to nearly a million and a half pounds. (*Cf.* example 3.)

Although in the above instance the company lost their case, and the "straight line" law has since been followed unchallenged in tramway cases, there is much that can be said for the sinking fund assumption for businesses sold freely as going concerns, the sale not being anticipated when the business was instituted. This is particularly clear in the case where there is only one asset, or where all the assets have about the same date of commencement and expiry. For then, if a "straight line" law is assumed, it is easy to see that the seller will be at a disadvantage owing to there having been more capital locked up in his part of the tenancy than in the buyer's. Thus if a mine were purchased on an estimated capacity of 1,000,000 tons for a price

of £200,000 (*i.e.*, *4s.* a ton), and if the mine changed hands when half the capacity had been won, the price on a "straight line" basis would be £100,000. But this would be unfair to the sellers, who have had to pay interest on the £200,000 throughout their period, whilst the buyers obtain the same advantage at only half the cost as regards interest on purchase price.

One objection to the sinking fund method of estimating interim depreciation is that its computation depends upon the rate of interest assumed, and this forms another possible source of dispute in the event of a sale. But there is no question that the sinking fund is an economically sound method (and the only equitable one as between the different years) of providing for total depreciation, and such provision must necessarily vary with the rate of interest, since it is a method of preparing by payments starting *now* for a future eventuality. If therefore it is sound for total depreciation and provides for all necessary replacements, it should be acceptable for interim valuations, *i.e.*, if it satisfies the engineer and the economist, the accountant and business man should fall into line.

The chief exception to this would appear to be that already mentioned, where the sale is anticipated beforehand, and the method of computation is already determined. Thus if a company is granted the monopoly of electricity supply or transport for a period of, say, fifteen years, the concern then to be purchased for the value of its physical assets, and if the company puts down plant whose life is thirty years, the method of valuation should be ascertained at the commencement. If this is to be the "straight line" method, the company will know it can receive only half the purchase price for the plant at the end of fifteen years, and must establish a fund accordingly. But it must not be forgotten that in all cases in which a "straight line" (or still more a reducing balance) formula is employed, the early customers are in effect overcharged and/or the early shareholders under-rewarded.

Special Formulæ.—When an undertaking is changing hands as a going concern, not only are the assets much more numerous and varied since they include all the items under the heading of goodwill, but also the valuation becomes much more difficult. In such cases it is important to realise that many things have happened since the undertaking was first set on foot, besides wear and tear and such-like obvious physical depreciation. The original equipment may have included items now seen to

have been unnecessary, yet which it would have been too risky not to provide, public demand may have altered, and other factors may have come in which change the remaining earning power, and should therefore be allowed for in any depreciation estimate.

Whilst one or other of the three formulæ given above is usually followed for the items to which it is applicable, there are a number of more complex formulæ which may be employed in particular instances. Such cases lie somewhat outside the scope of the present work,* since they concern not so much the engineer as the business promoter and financier, but it may be said that the principle upon which they are based is usually that of reflecting the probable future earnings of the undertaking. The endeavour should then be to give the purchaser the prospect of as good a return on his money as would be given by other investments of a similar nature and at the same time. It is not always easy to translate this principle into quantitative terms, but the chief guide in all such cases must be the average public estimate of the position and prospects of the undertaking, as shown by the price at which its shares were quoted before the transfer was mooted.

Tramway Terms.—When a community of people, in the shape of their central or local government, concede to a private company the right of undertaking any vital service which is monopolistic in character, *e.g.*, gas and electricity supply and tramways—the concession is usually hedged about by various restrictions designed to prevent the exploitation which is only too often the result of monopoly. In some cases these limitations chiefly concern the price to be charged, but in other cases, particularly when the service is one of vital importance to a large section of the community, the restrictions are in the direction of preventing the growth in private hands of a vested interest in what is felt should be a public concern. For this reason the concession is usually limited in time, at the end of which the community has the right to purchase the undertaking at the then value of the physical assets, exclusive of any allowance for past or future profits, or any compensation for compulsory sale—in a word, excluding "goodwill."

The above clause is embodied in the Tramways Act of 1870, and the terms of purchase are commonly known as "tramway

* The student should refer to such works as "Engineering Economics," by J. C. L. Fish (McGraw Hill Book Co.).

terms." It will be noticed that nothing is specified as to the mode of computing the "then value," but the usual method appears to be from the first cost less depreciation on a "straight line" basis up to the date of purchase. This still leaves to be settled the question of what is to be regarded as the normal life of each asset, and (in the case of composite items) when this life should commence, particularly if there have been large partial renewals charged to maintenance which might be construed as capital expenditure.

Wasting Assets: Income Tax Distinction.—Assets which experience the persistent loss in value defined as depreciation are frequently divided into two classes, (*a*) and (*b*), as defined below. This division has no great importance from the point of view of engineering calculations, since most of the assets there considered fall into class (*a*), and furthermore, there is no difference between the two as regards the provision which has to be made. The chief difference lies in the question of whether this provision can or cannot be deducted (for tax purposes) from the income received.

(*a*) Inherently wasting assets—defined as capital invested in the purchase of temporary sources of profit. These include plant, destructible buildings, ships and natural raw materials (minerals, etc.). It will be noted that the wasting is here a requisite in the processes of the concern. An income is only obtained at the cost of a definite loss in value in the plant, etc., and this cost must therefore be deducted before any income can be enjoyed.

(*b*) The assets in which the wasting is not inherent are defined as capital invested in the purchase of temporary interest in a permanent source of profit. These assets are generally not of the tangible portable variety, and are not represented by a definite corpus or fund which wastes in the process of earning income or profits. They are rather anticipations of future returns, and can be regarded as the purchase of rights to the enjoyment of advantage in later years.

A man who borrows money to purchase the lease of a building is in just as much need to lay by money each year, in order to repay the loan by the expiration of the lease, as the purchaser of a machine must lay by money against the date of its replacement. But the former asset, being a permanent source of profit (to whosoever has the use of it at the moment) is taxed each year as income, and the user of a lease must bear this

liability in mind when purchasing.* In the case of a machine, however, the profit only arises at the cost of the machine's wear; and a deduction from gross income to replace this wear is, in general, allowed before taxing the income arising from the use of such assets.

Unfortunately, whilst the above principle is recognised by the Exchequer, the application is often obscure, the amount and methods of making the allowances vary in different places, and the allowances themselves are often felt to be inadequate on one or other of the following grounds :—

Many items properly in class (*a*), *e.g.*, buildings (other than farm), minerals, etc., are excluded from any allowance.

The allowances are sometimes on the basis of a fixed percentage of the first cost (straight line law), but more usually on a percentage of the previous year's value (compound interest law); hence, unless the percentage which is allowed off the reducing balance is an ample one, the allowance becomes very small after a few years.

The allowances have in the past generally been only to cover wear and tear, and not full depreciation. In the Munitions of War Act, 1915, and Finance Act, 1916, however, the existence of "exceptional depreciation or obsolescence" of buildings and machinery is recognised, and in some cases a larger deduction can be claimed for items such as electrical machinery liable to be superseded.

* It will be noted that where an asset in class (*b*) includes also some inherently wasting element (*e.g.*, leasehold property requiring upkeep), the Exchequer recognises the double nature of the asset and there is a statutory allowance for repairs.

Worked Examples

1. The first cost of certain plant was £1,725. If depreciation is assumed to follow the reducing balance or compound interest law, 10 per cent. being written off each year from the value at the beginning of that year, what will be the valuation after ten years, and how soon (approximately) will it be worth one-tenth of its first cost?

$A = P(1 - i)^L = £1{,}725 \times (1 - 0{\cdot}1)^{10} = £1{,}725 \times (0{\cdot}9)^{10} =$ £601 9*s*.

Also if $172{\cdot}5 = 1{,}725 \times (0{\cdot}9)^L \quad L = \dfrac{\log 0{\cdot}1}{\log 0{\cdot}9} = 21{\cdot}9$,

i.e., 22 years approximately.

2. The first cost of a motor car was £342, and four years later it was valued at £140. By how much per cent. per annum had its value diminished, assuming that the decrease followed a compound interest law?

$$A = P(1 - i)^L, \text{ or } 140 = 342(1 - i)^4$$

$$\therefore 1 - i = \sqrt[4]{\frac{140}{342}} = 0{\cdot}80 \quad \therefore i = 0{\cdot}20,$$

i.e., 20 per cent. per annum.

3. Plant having a first cost of £2,000 has an estimated salvage value of £200 at the end of a useful life of twenty years. What would be the valuation half-way through its life (a) if depreciation is estimated on a straight line basis; (b) if depreciation is estimated on a reducing balance basis; (c) if depreciation is estimated on a sinking fund basis at 6 per cent. compounded annually.

Total depreciation, or wearing value = £1,800 in 20 years.

(a) Valuation after 10 years $= £2{,}000 - £1{,}800 \times \dfrac{10}{20} = £1{,}100$.

(b) Depreciation rate $i = 1 - \sqrt[L]{\dfrac{V}{C}} = 1 - \sqrt[20]{\dfrac{200}{2{,}000}} = 0{\cdot}109$,

i.e., 10·9 per cent.

Hence valuation after 10 years $= £2{,}000\,(1 - 0{\cdot}109)^{10} = £632$.

(c) Sinking fund deposit for 20-year fund $= \dfrac{1{,}800 \times 0{\cdot}06}{(1{\cdot}06)^{20} - 1} = 48{\cdot}93$.

In L years this would amount to (from equation 7) $A = D\frac{(1+i)^L - 1}{i}$.

Hence in 10 years this would amount to $48 \cdot 93 \; \frac{(1 \cdot 06)^{10} - 1}{0 \cdot 06} = 645$.

So that the valuation after 10 years = £2,000 − £645 = £1,355.

4. (*A.M.I.M.E.*, April, 1925.)—A method of providing for the rapid depreciation of a machine during its early years is to deduct a constant percentage from its value for the preceding year, and put this to the depreciation fund. Apply this method to a machine costing originally £1,000, estimated life fifteen years, and scrap value at the end of that period £200, and obtain the constant percentage written off. (Interest included.) Obtain also the amount in the depreciation fund at the end of ten years.

Presumably this means that each year a percentage is written off the loan and the interest payable on it, so that the depreciation deposits are regarded as not bearing interest. What is termed the depreciation "fund" is not an actual fund of money, but a paper credit whose value at any time is merely the numerical sum of the amounts paid into it.

$$A = P\,(1 - i)^L \text{ or } i = 1 - \sqrt[L]{\frac{A}{P}} = 1 - \sqrt[15]{\frac{200}{1{,}000}} = 0 \cdot 1017,$$

i.e., approximately 10·2 per cent.

Valuation at end of tenth year = £1,000 $(1 - 0 \cdot 1017)^{10}$ = £342.

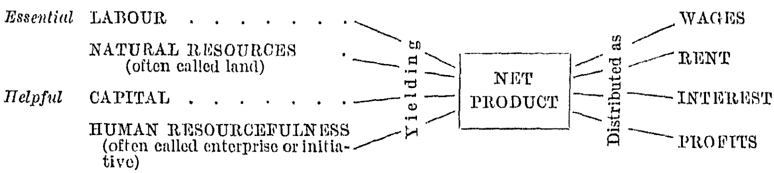
Hence credit in depreciation reserve at end of ten years

= £1,000 − £342 = £658.

CHAPTER IV

PRODUCTIVITY

Agents of Production.—It was seen in the beginning of Chapter I. that wealth production or the creation of economic utilities is the result of labour acting upon natural resources, and assisted by past accumulations in the form of capital, knowledge and initiative. Diagrammatically, it can be put as follows:—



Warning must again be made against taking the above distinctions too rigidly, and some of the difficulties of demarcation have already been touched upon (p. 6). Thus the distinguishing mark of natural resources is the fact that they are strictly limited in quantity or extent, so that the supply cannot respond to a change in the price or demand. But in a short period the same thing is true of the other agents, since they all take time to come into being, and even a big rise in the price offered will not at once greatly increase the supply. Thus in England, after the Great Plague, the supply of labour in proportion to the other means of production was so small that the most stringent laws could not prevent a considerable rise of wages, and for many years there was a spell of comparative prosperity for the unpropertied classes. Again, during the late war both capital and also many commodities showed similar temporary shortages and excessive rises in price.

Particularly is this the case with the last of the items mentioned above (human resourcefulness), which is in many respects similar in character to the item of natural resources. In each case there is an element of uniqueness, the price paid is a

scarcity price (due to the insufficient distribution of the best lands and the best brains), and it will be found that the toll levied by individual ingenuity in such forms as, for example, patent royalties, partakes of much of the character of economic rent. Such payments are therefore called by Professor Marshall "quasi-rents,"* but it must not be forgotten that this is essentially a short-period characteristic, and over a long period they are clearly distinguishable from true economic rents—the price paid for the inextensible gifts of unconscious nature. In the long run the quantity even of human initiative can be increased, so that the supply of all these agents, except that of nature, responds ultimately to the demand, and the price tends to become merely the supply price necessary to produce it.

For the purpose of a quantitative study of productivity, the above distinctions are not of great importance, since it is impossible to consider the effect of three or four variables at once. Drastic simplification is therefore necessary in order to keep the problem within two dimensions, and the usual plan is to merge all the agents concerned into two groups, one of which can be supposed fixed and primary, whilst the other is varied. Thus in agriculture the land and other fixed capital may constitute one group, whilst the labour, etc., constitutes the other, and it is then possible to speak of the productivity or fertility of a given plot of land for varying applications of labour. In other cases one can speak of the productivity of labour under varying conditions, or even of the "productivity" of a machine or a business.

Thus productivity is related to one or another agent or group of agents and regarded as its attribute, but it must never be forgotten that this division is quite artificial and has no sanction other than that of convenience. A highly complex process, such as any modern act of production, is the composite result of many factors, material and psychological; and although in theory certain agents may be considered as essential and others merely helpful, production in fact generally involves them all. Even in the relatively simple case of agriculture the harvest is a joint product of a number of factors—land, seed, labour, time, sun, etc.—each yielding nothing without the others.

* Another term of the same kind is "rent of ability." This is particularly applied to the toll levied by human ingenuity, whereas the term "quasi-rent" is more generally used for the net income from appliances of production already made.

Increasing and Diminishing Returns.—Having noted the several agents which may enter into any act of production, the next step is to study the effect upon the produce of varying the proportions of these agents. For this purpose the agents are divided into two groups as suggested above, and the classical treatment of this subject by writers in economics can best be illustrated by reference to agriculture, in which the land is regarded as the primary agent, and endowed with the quality of fertility or productivity.

James Mill was one of the first to attempt a quantitative statement of this fertility, and his method may be briefly summarised as follows :—consider a fixed plot of land and find what amount of produce could be obtained from it by various degrees of cultivation. Imagine the cultivation as made up of a number of small equal "doses" of labour, plus seed, etc., in suitable proportions.* Then if cultivation to the extent of, say, 40 of these doses resulted in a harvest of a certain size, whereas if 41 had been applied the harvest would have been slightly greater, that extra can be regarded as the "return" to the forty-first dose, and can be plotted as representing the fertility at this point.

Mathematically if at any point the degree of cultivation or work put into the plot of land is represented by W, the amount of produce or output by P, and each small dose and its return by δW and δP respectively, then the fertility at that point is represented by $\frac{\delta P}{\delta W}$, or $\frac{dP}{dW}$ when the unit dose is made infinitely small. In words, degree of fertility is defined as "the amount of produce which unit area yields from the expenditure of the last small portion of work" (Jevons). In more general terms productivity is thus exhibited as the rate of change of output with input, and the curve representing this quantity is usually plotted to a base of input to a fixed area. In Fig. 4 the area of the black column represents the difference between the harvest obtained from 40 and that obtained from 41 doses of input, *i.e.*, it represents the return (δP) to the 41st dose, and the height of the column (area divided by base) $=\frac{\delta P}{\delta W}$.

* The portions of cultivation are, of course, all applied simultaneously or as the occasion requires—the splitting up into doses is purely imaginary.

The general features of such curves are expressed in what are called the laws of increasing, diminishing, or non-proportional returns, the latter name being due to Gide.* That is to say, the curve of productivity generally shows a tendency to rise at first and fall later, but the proportions depend entirely upon the particular case, and in agriculture it is the diminishing returns which are chiefly in evidence. Thus if the area be fairly large and the unit dose very small (*e.g.*, one man's labour), it might be found that two men could produce more than twice as much, whilst a third man added more than the second (portion AB). But if the labour application is further increased, a point of maximum return per additional man is

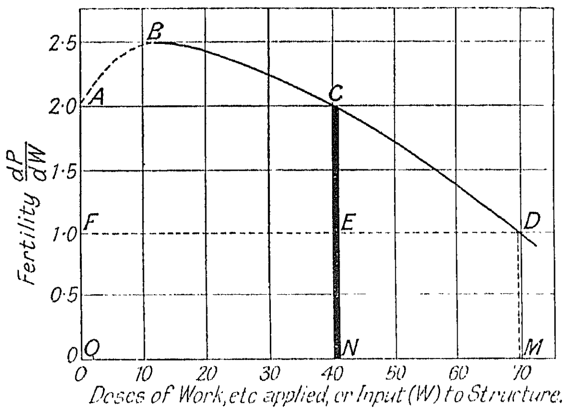


FIG. 4.—Fertility Curve.

very soon reached: "saturation" sets in, and higher degrees of cultivation show steadily diminishing returns (portion BD).

Industry, on the other hand, is generally said to illustrate the law of increasing returns, since additional applications of capital and labour, in general, show an increasing rate of return; but as this can hardly continue indefinitely, a point of maximum return can be considered to exist here also, although not reached nearly so soon. Strictly it is hardly possible to enunciate a law about anything so vague as "industry," since it is not clear to what the doses of labour, etc., are applied. Moreover, there has frequently been a confusion in the past between the law of returns (usually diminishing) which occurs when one agent, such as land, is kept

* "The law of maximum returns" would be a somewhat more definite title.

constant and the other varied, and the law of economy of large-scale production (usually increasing) which occurs when all the agents are increased simultaneously.

Another confusion has been created by the fact that some writers on economics prefer to define productivity from its efficiency ratio, output divided by input $\left(\frac{P}{W}\right)$, rather than from its rate-of-change value $\left(\frac{dP}{dW}\right)$. The present writer has applied the term "fertility" only to the latter definition, which has been used throughout this chapter, and which has a number of advantages, discussed below. The chief advantage of the efficiency definition is that it paves the way to a definition of economy. (See next chapter.)

It will be seen that there are many difficulties in the way of the enumeration of a rigid law of returns, and many authorities prefer to treat the subject descriptively rather than quantitatively. Perhaps the utmost that can be said is that throughout all production the curve seems to follow the same general shape, differently emphasised in different cases; and, in Professor Marshall's words, "While the part which nature plays in production shows a tendency to diminishing return, the part which man plays shows a tendency to increasing return."

Useful Conclusions: Economic Rent.—Several interesting points can be illustrated by means of the above method of defining fertility. Originally expressed only in *quantities* of input and output, it has been applied by later economists to show *values*, in which case both the input and output can be in the same unit (*e.g.*, in pounds' worth of labour and harvest), and the ordinates of the curve are then pure numbers. If cultivation is carried on so intensively that the point D is reached (ordinate MD = unity), the return per additional pound expended only equals £1, and cultivation carried beyond this point will show a loss on any additional expenditure (though not on the whole enterprise).

An increment of labour put in at the point D is therefore called the marginal dose, and the return MD, which only just repays the input, is called the marginal return.* Land which is cultivated up to this margin is yielding the utmost that can

* A very good example of this "marginal dose" occurs in connection with the economics of power factor improvement; see p. 257.

profitably be extracted from it, and for a farmer who has a fixed plot to till this represents the limit to which he can economically go. In a similar way the return to any particular dose, say the 41st pound (*i.e.*, the column *NC*), can be split up into two portions, one of which *NE* pays the cost of the dose, whilst the remainder *EC* is a surplus available for rent and profits.

It will further be noticed that, whatever the degree of cultivation considered, the area of the curve up to this point represents the total output obtained from the enterprise. For since the area of each column represents the extra output due to the dose just applied, the total area up to any point must represent the sum of all these returns, *i.e.*, the total output. This total can be considered as made up of two portions—the rectangle *FDMO* representing the amount necessary to pay expenses, and the area *FACD* representing surplus.

The diagram is also useful in illustrating what is called the "law of rent." Taking the world over, there will be some plot of land so inaccessible and so infertile that the curve never rises above the line *FD*. That is to say, even when cultivated in the best possible proportions, the value of the yield only just equals the cost of the labour, etc., expended (*sans* rent). The poorest of such plots which is actually being tilled at any instant is said to be on the margin of cultivation; and any better plot will give a yield which (priced at figures which just keep the marginal enterprises solvent) gives a surplus for rent and profits.*

The area above the line *FD*, in all land which is better than the marginal, is called "economic rent." Whatever the system of land ownership, this will exist (even though not apparent) whenever there are differences in the quality or accessibility of the land. Economic rent can therefore be said to subsist in or arise out of qualities in the land itself, and is a measure of the differential advantage of the more productive plots over the poorest land cultivated. If the land is owned by the actual cultivator he will reap this advantage himself, but if it is hired, the landlord will naturally endeavour to extract as much as possible of the surplus in the form of an actual rent. Such a

* Thus rent is often said to be no component of the price of wheat, the latter being such as just to make profitable the cultivation of the rent-free margins. This, however, refers rather to the effect of particular rents on the particular wheat grown on that land—naturally the total rent of wheat lands has to be paid for out of the total price paid for wheat. It is the *destination* of the rent that does not affect the wheat price.

rent might then be regarded as the equipoise necessary to make the same quantity of labour give the same net product under differing conditions.

Law of Large-Scale Enterprise.—If land were a perfectly uniform, and what is called later homogeneous, structure, it is evident that with a certain product resulting from the labour of one man on one acre, just double would result from two men working on two acres. But when a large farm is considered as a whole this direct proportionality does not quite apply, and it still less applies to more complex structures, such as machines or businesses, where two duplicates if rolled into one would not usually require double the input in order to give double the output of one.

The only question which has so far been considered, and the only one which can arise in connection with a homogeneous structure, is that of the most suitable *proportions* of the agents when one group is varied and the other kept constant; and the law of maximum returns can properly only be said to refer to this. With non-homogeneous structures there is another question, namely, as to the most suitable overall *size* of an undertaking when all the agents are varied simultaneously. The latter is distinct but closely related to the former, and appears to follow similar laws, which may be called the law of increasing, diminishing, or maximum economy of large-scale enterprise. Thus with any given ratio of W to S a proportionate increase in both of them may produce a more or less than proportionate increase in P.

Whatever the type of undertaking, unless it is strictly homogeneous, increasing economy is usual until a very big size is reached, *i.e.*, the point of maximum economy (of large-scale enterprise) will be late in arriving. But the general shape of the curve is similar to that of the two productivity curves already considered, and it is not surprising that they have often been all three coupled together without distinction.

The question of the most suitable size of an undertaking when all the agents are varied simultaneously can hardly be said to be the concern of the engineer as such, but rather that of the financier or *entrepreneur*, since the desideratum is then the most profitable enterprise rather than the most economical service. It will therefore not be considered in detail in this book, which will be chiefly concerned with problems in which the output or service required is fixed and known beforehand.

Engineering Examples.—There are many examples in engineering practice of what might be called increasing and diminishing returns. Thus in studying the effect of illumination upon the output of a factory, if the lighting is very dim a small expenditure in this direction will yield very big returns, but as the illumination is still further increased there will come a time when its return only just covers its cost, and this might be called the marginal dose. Beyond this it would be uneconomical to increase the illumination, since any slight increase in output which resulted would not pay for the cost of the extra light. In a similar way, when obtaining flux from an electro-magnet, there may be said to be first increasing and then decreasing returns to the ampere-turns applied, since the permeability first rises to a maximum and then falls again. (The actual curve used to illustrate productivity (Fig. 4) is a portion of the graph plotting rate of change of flux with magnetising ampere-turns for a piece of wrought iron.)

Another example, in which the agents concerned can easily be divided into two groups comparable with land and labour, is that of a motor supplied with electrical power and giving a mechanical output. If at first the power put in is very small, it will do little more than supply the constant losses, so that an increase in the input will here yield greatly increased returns of useful output. But if the process is continued a point of diminishing returns is ultimately reached, owing to some of the losses being proportional to the square of the input.

Many other examples might be given, but in most cases they simply resolve themselves into the commonsense of correct proportionality—the most efficient result is obtained when the best balance is reached between the different agents contributing towards that result. The difficulty in these cases is not to recognise the principle, but to express it in quantitative terms and apply it usefully. As was seen above, there is far from being any rigid law of nature which can be called increasing and diminishing returns: at most, it is no more than a general tendency discernible in a number of parallel cases.

Any wholesale generalisation about the "laws" of production in industry or manufacture is therefore likely to yield little useful result, and will not be attempted here. Endeavour will rather be made to isolate a single structure or portion of a structure and to examine the economy of the performance of its particular service. It can then be treated, just as Mill

treated his portion of land, with successive "doses" of labour or energy; but in this case the efficiency conception of productivity, *i.e.*, the ratio of output to input, will be found more useful than the rate-of-change conception, since the former can be developed into a criterion of economy. Before proceeding to this consideration, it will be necessary to establish a suitable dividing line between the structure and the elements operating it.

CHAPTER V

ENGINEERING APPLICATIONS: BASIS AND METHOD

Division of Agents in Engineering Problems.—It will be recalled that the starting point of the foregoing study of productivity was to divide the agents of production in any case into two groups. One of these is regarded as fixed and primary—part of the data of the problem—to which is applied varying proportions of the second group. The latter therefore constitutes the "independent variable" of the problem, whilst the product which results is the "dependent variable."

In the case of agriculture the main agents of production are land and labour. With the former can be grouped all the fixed aids to production (rivers, buildings, etc.), whether natural or artificial; with the labour is grouped the seed, the hand-operated tools and all other items which need to vary with the amount of the labour application. It is then possible to study the fertility of the land group for varying applications of the labour group, but it must not be forgotten that this division into two is really arbitrary and for convenience only; strictly the harvest is the corporate product of all the helpful agents.

In engineering problems the same sort of simplification can often be performed with advantage; when a machine or equipment of any kind or even a business can be operated by varying degrees of labour to produce varying amounts of product this procedure can be followed, and it is then possible to study the "productivity" of the machine or equipment. It is even possible to do the same thing with plant, such as an electric motor, in which no human labour as such is employed, but in which the electrical energy consumed can be considered as taking its place.

Hence in any engineering problem having a range of possible solutions, such as finding the most economical way of obtaining a given service or operating a given machine, the usual method will be to divide the agents concerned into two groups; namely, the instrument, appliance or equipment by means of which the operation is carried out (or facilitated), and the work or

energy which actually does it. In some cases the terms "capital" and "labour" will describe the two groups, but a more useful conception is to think of the former as a fixed structure or vessel (S), into which is poured varying quantities of the working component (W).*

With this distinction in mind it will be possible to draw a rough dividing line between the expenses under S and those under W, *i.e.*, between those involved in providing, maintaining and replacing the structure, and those involved in operating it. Since the distinction is primarily one of use rather than kind, the division will usually be made between the expenses which go on from year to year whether the service is performed or not, and those which are destructively consumed in the operation itself and cease when the service is suspended. The table given below will serve to emphasise this distinction, and the examples given will show where the line might be drawn in particular instances. (*Cf.* also the table on p. 102.)

These examples will also illustrate the point previously emphasised that it is difficult to draw a hard and fast line, since the division is largely an arbitrary one. Just as biologists distinguish between structure and function, and divide medical science into anatomy and physiology for convenience in study, so the expenses of a service can be split up into those which are periodic and relate to the structure, and those which are dependent upon function or output. Actually nature knows no such rigid lines, and many of the expenses merge and are difficult to allocate. Thus depreciation usually depends partly on time and partly on wear (*i.e.*, output), and cannot strictly be put into either group. But when obsolescence is also included, the time element becomes the more important; and experience will usually enable the life of any plant to be roughly estimated in years. The depreciation provision then becomes a simple annual item, which is a great convenience, as it can be grouped with the interest, under S. In some cases an item will have to be split up between the two groups, as in the expenses of a power station. Here a considerable part of the labour, etc.,

* It will be noted that in a large number of cases the elements composing the second group are of the nature and in the units of work, consisting as they do of human labour, coal, mechanical work and electrical energy. In physics, work and energy are synonymous terms, whether measured in foot-pounds, horse-power hours or kilowatt hours, whilst the heating or chemical energy represented by the calorie or a ton of coal are of the same kind and can be expressed in the same units. In the vaguer terms of industry the unit of labour or work, such as the man-power-week, is also of the same nature, and in purely physical manifestations it can be similarly expressed.

will come under the S group, being a function of the plant and practically independent of output.

Division of Agents in the Performance of an Engineering Service

Symbol	S	W
Dividing line is drawn between items whose expenses is primarily proportional to - - -	Time	Output.
Expenses are therefore -	Standing or overhead charges	Working or running charges.
And are required to - -	Provide, renew and maintain the structure	Operate the structure.
The agents themselves are usually in the nature of	Capital, structure or plant	Labour, work or energy.
Examples in the case of the following services :		
Transmission of electrical energy	Cable (interest and depreciation)	Electrical energy.
Mechanical power from electric motor	Motor and equipment (interest and depreciation)	Electrical energy.
Illumination by electrical means	Lamps (depreciation)	Electrical energy.
Electrical energy from power station	Steam and generating plant (interest and depreciation, staff, etc.)	Coal, oil, etc.

The Balance: Physical and Economic Limits: Regulation.—Having established a dividing line between the two groups of agents entering into any process, the next step is to find what balance between the two groups gives the most satisfactory result. If the S group can be regarded as fixed and constant, the problem becomes—what quantity of the W group will give the best result? and this is a problem which can be settled in either of two ways—physical and economic. Moreover, the two solutions are entirely independent, since they relate to quite different criteria, so that every likely case needs to be considered on both sets of grounds.

Continuing the metaphor suggested above, the structure

or vessel S may have an actual capacity which is a more or less definite physical limit beyond which it will give way (*i.e.*, burn out or break down). On the other hand, it may be adjudged uneconomical to fill it beyond a certain point, because greater leakage takes place (decreased efficiency), although the maximum capacity has not been reached. It will be noticed that the first mentioned or physical limit is a more or less compulsory "dead stop," whereas the second is only a position or area of maximum economy round about which it is advisable to work.

In the case of agriculture there is a certain degree of cultivation which, applied to any given plot of land, will give the best return to the labour put in. If the input is made to include also the cost of the land itself, this becomes an economic solution to the problem. But if this input density were greatly exceeded there might come a point at which the (physical) limit of space was reached and overcrowding occurred.

In the realm of electrical engineering, owing to the compactness of most of the apparatus employed, the physical limit is usually reached first, since the question of heat dissipation then becomes paramount. There are, however, a number of cases in which the losses increase with the density of loading to such an extent that the economic minimum is reached before the physical limit, and one of the main objects of this book is to lay down the principles and methods of solution for such cases.

It seems remarkable that engineering, whose aim is to satisfy certain human requirements with the least expenditure of human energy, should make so little apparent use of economics; but this is due to the early occurrence of physical limits (such as the heat capacity mentioned above, physical strength and the like) and to the fact that in many cases the losses or maintenance expenses are not an important item of the cost or cannot be materially reduced by an increase in the capital outlay. Thus it comes about that engineering is regarded as primarily an application of physics, and the only economic tenet underlying most engineering design and selection is the conviction that lowest costs will be reached by using the minimum equipment necessary to satisfy physical requirements.

In the author's opinion the above assumption is by no means always justified, and a large number of cases should be examined on economic as well as on physical grounds. In general it may be said that wherever the working expenses (W) are

large compared with the structural expenses (S), and wherever the two groups are reciprocally related (so that increased expenditure on equipment would save running costs), there exists a suitable case for economic investigation, provided that the range of action is large enough to reach some point of saturation or diminishing returns. A glance at the table on p. 102 will give some idea of the likely cases occurring in electrical engineering.

In addition to the physical and the economic sets of criteria, there is a third set which in some cases may be the determining factor in fixing the size or quality of plant. This is the item known as regulation or voltage drop, and it will sometimes be found that a size of machine which was satisfactory from the point of view of heating and also from the strictly economic aspect would be too small on grounds of regulation. It is difficult to find a parallel to this in the case of agriculture, but it is possible that in addition to economic and physical considerations in fixing the number of men working on a given area there might be some legal regulation which would correspond to some extent. A better illustration, taken in this case from another branch of engineering, might arise if a company were erecting a structure such as a suspension bridge for private profit collected by a toll on the users. A certain size would be essential for strength; there would also be a most economical size, since a very small bridge would deter traffic, whilst a very large one would not bring in a proportionate return; and lastly, there might be a certain size necessary to avoid the annoyance of excessive swaying or deflection, or a certain minimum width for convenience in use. This last factor, which need not be directly economic in character, but is rather a question of satisfactory service to the public (and sometimes of statutory "regulations"), corresponds roughly to the regulation limit in electrical apparatus.

Types of Structure.—In electrical work there are three main kinds of losses, copper, iron and frictional, and it is therefore convenient to group the structures which may be employed in any electrical service into three classes, according to whether they have one, two or all three possible kinds of loss. These are:—

Single-loss structures, having (substantially) only copper or iron loss.

Examples: Cables and choking coils.

Two-loss structures, having both copper and iron loss.

Examples: Transformers and electro-magnets.

Three-loss structures, having copper, iron and frictional loss.

Examples: Generators and motors.

Structures of the first class may be said to be uni-dimensional or economically homogeneous with respect to the particular service in question, so that if every item in the performance is doubled, the product will be doubled, and the efficiency will be the same as before. A parallel case occurs with land, which, in respect of cultivation, may be said to be a homogeneous (*i.e.*, a uniform undifferentiated) type of structure, so that if a certain product resulted from one man working on a given area, just double would result from two men on twice the area. Hence the base of the curve in Fig. 4 (p. 77), instead of being the input to a fixed plot of land, could be called "input density," and the one plot then epitomises the whole of cultivation for soil of that quality.

Electrically the best examples are copper and steel, as conductors of current and flux respectively. Both of these may be said to be virtually homogeneous in the performance of these particular functions, so that a solution (in terms of densities) for a small conductor is frequently applicable to a big one.*

Complex structures, such as transformers and motors, having more than one kind of loss, are clearly in a different category, since neither the capacities nor the losses can be represented as linear functions; and if the frame and the output are both increased together (so as to keep the densities the same), the efficiency will not be the same but will be inherently better. Another way of putting it is to say that whilst two 5-h.p. motors side by side take twice the input to give twice the output of one, if they are amalgamated into one 10-h.p. machine a considerable saving is effected. With a cable, on the other hand, two identical cables can be amalgamated into one of twice the size, working at the same current

* It will be noted that in no case is the above strictly true. Even with land, although the yield to the actual labour applied is unaffected by neighbouring areas, in the farm as a whole there may be co-operation and transit arrangements, to say nothing of contagious weeds and diseases, which affect the strict proportionality. In the same way it may be said that whilst copper and steel worked at certain densities have certain properties and losses, there are other factors, such as cooling, eddy currents or demagnetisation by the ends, which are affected by contiguous areas.

density, without affecting the economic situation as regards current conduction, losses, etc. (see Appendix IV.).

Hence it will be found that in a few simple cases involving structures of the first class (of which cables are the only likely example), the solution is in the form of a current density, which is generally applicable to any section, large or small. In all other cases, and wherever the structure employed has more than one kind of loss, the solution is a particular one and only applicable to the actual size of structure considered.

Types of Problem.—Summarising the foregoing, it may be said that any act whose performance it is desired to study may conveniently be considered as made up of a machine or structure S, to which is applied a quantity of labour or energy W, so as to produce a certain quantity of product or service P. Further, it has been decided that the criterion of satisfactoriness of the performance from the economic point of view shall be the efficiency ratio, output divided by input (both expressed in money values). Treating the operation in this way, as though it were a definite entity, the input to it is the sum of the expenses under S and those under W, whilst the output is the value of the product or service obtained. The economy can then be expressed as:

$$\frac{\text{Value of output (product or service)}}{\text{Cost of input} = \begin{cases}\text{cost of structure plus} \\ \text{cost of working}\end{cases}} \quad \text{or} \quad \frac{P}{S+W},$$

the symbols representing money values reckoned on some convenient basis.

Since there are three quantities in the above expression, there are actually four possible problems which might demand solution, namely those in which P is fixed, S fixed, W fixed, or all three variable. As regards the last named, that of finding what is the total size of an undertaking (all the production factors being variable) which will yield maximum profit, this is the business man's or *entrepreneur's* problem rather than the engineer's, and will not be treated here. Another problem which can be summarily dismissed is that in which W is fixed, as this is rarely or never likely to occur in practice.

Of the two problems which remain, those of S fixed and those of P fixed, so far it has always been the S element that has been assumed to be known beforehand—given a fixed plot of land or piece of copper or steel, what is the most economical way of working it? But in engineering practice it is much more often

the output, P, which is known beforehand—given that a certain service is required, what is the most economical way of obtaining it? The former type of problem might arise if the owner of a cable already laid wished to know the most economical current to transmit; the latter type occurs when a fixed current has to be sent and it is desired to know the most economical section to lay down. It will be noticed that in the latter case it is not necessary to work out the maximum value of the ratio $\frac{P}{S + W}$, but merely to find the minimum value of $S + W$, since P is fixed. Maximum economy becomes in this case minimum total cost.

A further point to note is that with a homogeneous or single-loss structure these two types of problem may be said to coalesce, since in such cases the value of the economy depends only on the ratio of the three components and not upon their magnitudes. The components can therefore all be enlarged or reduced at will provided their proportions are unaltered, so that whichever way such a problem is stated, it can be solved either by assuming fixed structure or by assuming fixed service—whichever is more convenient. An example of this is the case just mentioned, that of Kelvin's law for cable economy, in which the same current density would be indicated whether the starting point were a fixed cable or a fixed current to be carried (see next chapter).

The primary problem of engineering economics may then be said to be that of discovering the least expensive method of performing some given service, and the great majority of the cases to be dealt with can be comprised under this description. Such problems usually concern the economic choice of structure—alternative plants to obtain a given result—and the alternatives may refer to :—

Different sizes of the same type of plant—with correspondingly different efficiencies or lives.

Different types—*e.g.*, A.C. instead of D.C., high or low speed, motor generator or converter.

Different qualities—*e.g.*, high or low efficiency, use of special steel, insulation, etc.

There is a sub-division of this problem of economic choice which may be helpful. The individual purchaser of one or more items of plant is usually considerably restricted in his choice. Physical limits largely condition his range of action, and when

these are all satisfied, the possible alternatives which can be selected on economic grounds are very few. Probably he is left with not more than two or three alternative quotations, equally satisfactory in all other directions but differing economically owing to price, efficiency or what not. The problem which then arises can be called the choice between the specific alternatives, and it will be clear that, in solving it, all that is necessary is to add up the costs (on some suitable basis) incurred in each of the alternatives and then to compare the totals.

On the other hand, purchasers in bulk or designers of machines for their own use, are not limited to a rigid quotation or price list. If, for example, it is suggested that a lower current or flux density might be economically sound, this could be obtained by using more copper and iron, *i.e.*, a larger machine, but it is not always necessary to choose between actual listed sizes. In this case there is a range of alternatives available, and the problem is to find how far (if at all) a certain tendency should be followed.

Problems of the latter kind, *i.e.*, those involved in finding the most economical point in a range of alternatives, can only be solved by plotting the range and finding the point of maximum economy or minimum costs on the curve (*cf.* Figs. 5 and 7, pp. 107 and 144), or by differentiating if the values can be algebraically expressed. It will be clear that these problems are more difficult, and their solution can be said to include that of the others. For this reason they have in several cases been chosen for illustrative purposes in the text of this work, but plenty of examples of the choice between specific alternatives can be found elsewhere in the text, and also in the worked questions at the end of each chapter.

Bases of Economic Comparison.—Having defined the relative economy of any service as the ratio $\frac{\text{Value of output } (P)}{\text{Cost of input } (S + W)}$, the question arises as to how these various costs are to be computed so as to afford the most useful economic comparison. It is evident that they must all be reduced to some common denominator, and as the two groups S and W have been chosen so as to be functions respectively of time and output or service, it is evident that the most convenient basis of comparison will be either a time or a service unit.

In the great majority of cases it will be found best to treat them on a time basis, and the main items of cost incurred in

obtaining a given service can usually be expressed as functions of time in the following manner :—

Costs under S { The interest on the capital cost of the structure employed is necessarily an annual item.
The depreciation can be similarly expressed if the useful life (in years) is known or can be assumed.

Costs under W { The material, labour or energy used and wasted is a function of time multiplied by the load factor or by the annual hours of service.
The maintenance and repairs can usually be similarly expressed.

In all such cases the basis of comparison will be a time unit, either (*a*) annual cost, or (*b*) total capitalised cost (for the whole useful life). In the few cases for which a time basis is unsuited, a service unit (*c*) can be made the basis of comparison.

✓ Basis (*a*). For this purpose the service is presumed to continue indefinitely, fresh plant being purchased whenever the old is worn out or obsolete, and each item of cost is expressed annually. This is the basis which will be found most convenient in the great majority of cases ; it has been used in all the examples so far, and it will be used in the remainder of this chapter and in most of the examples in this book. It will be noticed that most of the expenses outlined above occur naturally each year in approximately even sums, and the only difficulty lies in " annualising " the first cost and salvage values. As this method is so useful, it will be found convenient to use a distinctive word to indicate annual expenses, and the word " charge " has been used throughout the book with this meaning.

Basis (*b*). This method consists in summating the total expenses of the service over a fixed period of years, either the lifetime of the plant or the period for which the service is required. These expenses must be related to some date (present or future), so that any items which occur annually must be capitalised or expressed in terms of their present or future worth. This basis is particularly useful when there are a number of capital items referring to different periods and when there are no annual differences to consider. It can best be illustrated by actual examples (see end of chapter).

Basis (*c*). Cases may arise in which a time basis of any sort is unsuitable. Such cases occur wherever the interest is negligible in comparison with the depreciation, *i.e.*, where the struc-

tures employed have a relatively short life and become "worn out" by service rather than time. A typical example is that of illumination, since the lamp has a comparatively small first cost, but it has to be renewed with a frequency depending roughly on the amount of use. In such cases the basis of comparison should be a service unit rather than a time unit, *e.g.*, lumen-hours in the case of illumination.

Summarising the above, the three bases of comparison are:

(*a*) Cost per year, called the Annual Cost Basis.
(*b*) Cost per lifetime of plant, called the Capitalised Cost Basis.
(*c*) Cost per unit of output or service, called the Unit Cost Basis.

Mode of Procedure.—The mode of procedure in the solution of any economic problem varies very much, but the following general outline may be useful:—

1st. Determine what is the question at issue and what are the fixed data. In problems in which there is a range of possible alternatives, this question at issue will generally furnish the independent variable of the curve which is to be drawn or calculated.

2nd. Decide the basis on which the economic comparison is to be made—whether annual costs, capitalised values, or costs per unit of service.

3rd. Tabulate every item of expenditure which can possibly be affected by the question at issue. These must be computed on the basis decided upon, and should be expressed as constants or variables relative to the question at issue. Items which are clearly unaffected by this question may be omitted from the list.

4th. When the problem is one of choice between two or more specific alternatives, it is merely necessary to add up the economic items in each case (all expressed on the same basis) and compare the totals. When there is a range of possibilities, the usual plan is to plot the various costs to a base representing the question at issue, and so find the best point. If the items can be expressed by formulæ, an algebraic solution is possible by differentiating with respect to this question.

Schedule of Total Costs.—As a simple example of the most general type of economic question, let it be supposed that the

problem is to determine the cost of performing a fixed known service by means of structure I, the final object being to compare this structure with one or more alternatives, II, III, etc., for the performance of the same service. Since the structures are assumed to be similar in respect of the output or service they perform, they only differ economically, *i.e.*, in first cost, salvage values, life, etc.* These items in the case of structure I can be denoted by C_1, V_1 and L_1 respectively. (In the schedule below only structure I is considered, and the suffixes are omitted for the sake of clearness.)

The problem as stated above makes clear what is the object of the determination, and the next step is to decide what is to be the basis. The most usual basis is that of annual costs or charges, and it will be assumed that this is the one to be employed here. In the table below, an attempt is made to suggest the outline of a statement, on this basis, of the total cost of performing the service. For this purpose the annual expenses are divided into two groups, as suggested earlier in the chapter, namely, the standing or overhead charges, which are necessary to provide and maintain the structure in full working order, and the working or running charges necessary to operate it.

The standing charges can be divided up into those involved in owning the structure (capital charges) and those involved in maintaining it. The former can be split up again into hire charges or interest Ci, and renewal charges or depreciation $(C - V)d$, where d is the annual deposit necessary to realise unity in a sinking fund whose life coincides with the useful life (L) of the structure. ($d = \dfrac{i}{(1 + i)^L - 1}$ for end-of-year payment (see equation 7, p. 19, and Appendix II.). Also when V is proportional to C, or so small in comparison that it can be treated as proportional, the depreciation item can be written Cd', where d' is the equivalent annual deposit $\left(= d \times \dfrac{C - V}{C}\right)$, and the whole capital charge then becomes $C\,(i + d')$.

Maintenance charges can be sufficiently indicated by the headings rent, etc., shown on the chart, and can usually be resolved into items proportional to the first cost of the

* Items of efficiency and repairs are also economic items.

SCHEDULE OF CHARGES

Representing annual expense of performing a given service by means of structure I.

S — Standing (or overhead) charges.

- Capital charges.
  - Hire of structure (Interest) Ci
  - Renewal of structure (Depreciation) $(C - V)\, d = Cd'$
- Maintenance charges.
  - Rent
  - Rates Insurance, etc.
  - Permanent staff and upkeep

W — Working (or operating) charges.

- Useful materials and labour
- Inefficiency and running repairs

$C\,(i + d')$ $\qquad$ $\alpha C + \beta$ $\qquad$ γ $\qquad$ $\dfrac{\delta}{C}$

$$C\,(i + d' + \alpha) + \beta + \gamma + \frac{\delta}{C}$$

Total charge $T = F\,(C) + \text{Constant} + f\,(C)$.

equipment, and items which are constant relative to this first cost.

The working charges * are more difficult to outline, since they necessarily vary more with the type of service. Under this head will come the bulk of the raw material, labour, energy and what not, which is proportional to the output or service rendered. In many cases this can be split up into that which is transformed into useful output and that which is wasted through imperfections in the structure. The last named can be expected to be less with a greater expenditure on structure cost, and the two might then be tentatively expressed as $\gamma + \frac{\delta}{C}$, where γ and δ are constants.

When the object is merely to find the cost of the service by structure I with a view to comparison with one or more alternative structures, there is no need to express the charges as functions of C, and it is only necessary to fill in the actual figures and total them up for each structure in turn. When it is a problem to determine the economical point in a range, endeavour should be made to evaluate the constants a and δ. The total charge T can then be expressed as the sum of two functions of C, the second one being possibly an inverse relationship. When the latter occurs, an economic solution becomes possible —thus by performing the above summary for the whole range of alternatives a curve can be plotted or the constants evaluated, and in the latter case the last expression on the above chart can be differentiated with respect to C, giving $\frac{dT}{dC} = F'(C) - f'(C)$, which will be zero (minimum total cost), when $F'(C) = f'(C)$.

The physical meaning of the above would be that if the first function $F(C)$ represented the charges dependent on capital, and the function $f(C)$ those inversely dependent, the minimum charge would occur when the rates of change of the two groups (with respect to C) were equal and opposite. At this point, sometimes called the point of "cost equilibrium," a small increment of capital expenditure will save just as much per annum in inefficiency charges as it incurs in capital charges.

* The above divisions will not necessarily coincide with any of the usual cost groupings. Thus the term "working expenses," as generally employed, includes everything except the capital charges and profits.

It is needless to say that the above is merely a simplified outline intended as a guide, and probably no one actual case will exactly fit this framework. In many cases the constants a and δ, even where they are empirically obtainable, will be in the form of powers of C rather than multiples. In many cases also the independent variable will be not the whole but some part of C, or even some item such as frame size or efficiency which is related to C.

Worked Examples

The best examples on the work of this chapter occur in the actual electrical problems which are treated subsequently. One or two exercises in first principles are, however, given below, and in one case alternative methods of solution are shown.

1. For performing a given service the following alternatives are available :—

Wooden Structure : first cost, £400 ; salvage value, £50 ; life, 15 years ; annual maintenance, £30.

Concrete Structure : first cost, £1,000 ; salvage value, zero ; life, indefinitely long ; annual maintenance, £10.

Find which of the two will be cheaper: (*a*) if the service is required for thirty years ; (*b*) if it is required indefinitely. Reckon interest at 5 per cent. per annum.

Basis of Comparison : Annual Costs or "Charges"

Wooden Structure—

		£ per annum.
Interest on first cost (C)	$400 \times 0{\cdot}05$	$= 20$
Depreciation on ($C - V$)	$350 \dfrac{0{\cdot}05}{(1{\cdot}05)^{15} - 1}$	$= 16{\cdot}2$
Maintenance		30
Total		66·2

i.e., £66 4*s*. per annum.

Concrete Structure—

Interest . . .	$1{,}000 \times 0{\cdot}05$	$= 50$
Maintenance		10
Total		60

i.e., £6 4*s*. per annum less than the wooden one.

If the service is required for an indefinite period the concrete structure will incur only the two charges shown (case *b*), but if it is only required for thirty years and the structure is then useless, the following depreciation charge must be added :—

$$1{,}000 \times \frac{0{\cdot}05}{(1{\cdot}05)^{30} - 1} = 15{\cdot}05$$

Making a total of . . . . 75·05

i.e., £75 1*s*. per annum,

ɔh is £8 17*s*. per annum more than the wooden one (case *a*).

Alternative Basis of Comparison: Capitalised Cost or Present Worth [Portion (*a*) only]. Service for Thirty Years.

Present worth of first wooden structure £400

Present worth of second wooden structure—less for salvage value of first (15 years ahead):—

(From equation 3) $P = \frac{A}{(1+i)^L} = \frac{400 - 50}{(1\cdot05)^{15}}$ = £168

Less for salvage value of second (30 years ahead):—

$- \frac{£50}{(1\cdot05)^{30}}$ = − £11

Present worth of 30 end-of-year sums (for maintenance) of £30 (equation 8)—

$$P = D\frac{(1+i)^L - 1}{i\,(1+i)^L} = £30\,\frac{(1\cdot05)^{30} - 1}{0\cdot05\,(1\cdot05)^{30}}$$ = £461

Net total . . . £1,018

Hence, to obtain 30 years' service, using the wooden structure requires a present lump sum of the above amount.

Present worth of concrete structure £1,000

Present worth of 30 maintenance sums of £10 (one-third of previous amount) £154

Total . . . £1,154

Hence the concrete structure costs more by £136, which is of course the present worth of the annual difference (£8 17*s.*) found by the previous method.

2. In a certain installation required to meet a steadily growing demand, figures are obtained for two alternative plans as shown below. Determine which of the two will be more economical for the twenty-five years' service, assuming that there are no annual differences to consider, and reckoning interest at 6 per cent. per annum:—

Plan 1.—Installation *A* costs £2,000 and serves for five years; after which it must be replaced by Installation *B* at a cost of £3,000, which serves for eight years; after which it must be replaced by Installation *C* at a cost of £5,000, which serves for a further twelve years.

Plan 2.—Installation *X* costs £3,500, and serves for ten years; after which it must be replaced by Installation *Y* at a cost of £6,500, which serves for fifteen years.

Plan 1—

Present worth of Installation A			$= £2,000$
,, ,, ,,	$B =$	$\dfrac{£3,000}{(1{\cdot}06)^5}$	$= £2,242$
,, ,, ,,	$C =$	$\dfrac{£5,000}{(1{\cdot}06)^{13}}$	$= £2,344$
Total . . .			£6,586

Plan 2—

Present worth of Installation X			$= £3,500$
,, ,, ,,	$Y =$	$\dfrac{£6,500}{(1{\cdot}06)^{10}}$	$= £3,630$
Total . . .			£7,130

Hence Plan 1 will be the cheaper.

3. In putting down a street distribution system, 6 ducts are required now, and another 3 will be needed later on, the problem being to find whether it will pay to lay the whole 9 ducts straight away. The costs of laying, including opening up and making good, are as follows :—

For a 6-duct line 32*s*. per yard run
,, 9 ,, ,, 45*s*. ,, ,,
,, 3 ,, ,, 22*s*. ,, ,,

Taking all capital charges as 8 per cent. per annum, find how soon the new ducts must be needed in order to justify putting them down now.*

The extra cost now would be 13*s*. per yard run.
The extra cost later would be 22*s*. per yard run.

Hence the question is, in how many years will 13*s*. amount to 22*s*. at 8 per cent. compound interest ?

$$A = P\,(1 + i)^L \text{ or } L = \frac{\log \dfrac{A}{P}}{\log\,(1 + i)} = \frac{\log \dfrac{22}{13}}{\log 1{\cdot}08} = 7 \text{ (almost).}$$

Hence if the ducts are required before the end of seven years they should be laid down now.

* This example is taken from Mr. F. Gill's address to the London Student's Section, *Journal I.E.E.*, 1923, Vol. 61, p. 791.

PART II

CHOICE OF PLANT

TYPICAL ELECTRICAL SERVICES,

for which economic choice of plant is possible.

Dealt with in Chapters.	Service rendered.	Structure employed.	Size, etc.	Annual Cost of Energy wasted in Structure as a Ratio of Annual Cost of Structure.	Cost of Energy entering Structure as a Percentage of the Total Cost of Service.	Cost of Structure as a Percentage of the Total Cost of Service.
VI. (p. 105)	Transmission .	Single Core Paper Cable	650 Volts D.C., 1,600 *A* per sq. in. Density	3·1 times	98·2 %	1·8 %
VII. (p. 119)	Motive power .	Squirrel Cage Induction Motor.	40 h.p. 1,000 R.P.M. .	6·3 ,,	98·2 ,,	1·8 ,,
IX. (p. 156)	Transformation	Single Phase Transformer	100 kVA, 50 periods .	1·8 ,,	99·1 ,,	0·9 ,,
X.	Illumination .	Vacuum Tungsten Lamp	60 watt . . .	6·3 ,,	87·0 ,,	13·0 ,,

Note.—The cost figures given refer to plant chosen by normal present-day standards, based generally on heating limits. They therefore give some indication of the probable room for improvement through economic choice. In the first three cases energy is taken at 1*d*. and the plant is assumed to have a life of twenty years, zero salvage value, with interest at 6 per cent. Service is at full load for 8 × 300 hours a year. In the last case energy is at 3*d*., the lamp at 2*s*. 3*d*., and the luminous efficiency is taken as 6 per cent. In no case is supplementary gear included or any expenses for housing, laying or installation. The above information is shown in diagrammatic form in the frontispiece.

CHAPTER VI

KELVIN'S LAW FOR CABLES

Statement and Proof.—The simplest application of the principle mentioned in the last chapter, of an economic balance between the expenditure on structure and the expenditure on operation, is that of Kelvin's law for cables. It is a rule for finding the most economical cross-section to employ in order to transmit a known steady current, and as originally propounded by Lord Kelvin (then Sir William Thomson) to the British Association in 1881, it concerned only bare conductors in which the cost of drawing the wire was either negligible or else proportional to the cross-sectional area.

It will be seen that this is that rare case, a perfectly homogeneous structure, so that a solution of the problem to carry 20 amperes would be just twice that to carry 10. Moreover, it will be seen that of the two variable costs in question, namely the cost of the conductor and the cost of the energy wasted in the cable, one is directly proportional to the cross-sectional area and the other is inversely proportional.

The rule was soon modified to meet the case of an insulated cable in which the cost of the insulation, etc., could be divided into two portions, the one constant and the other proportional to the area ; and as generally stated, the rule now is :—

The most economical section to employ is that which makes the annual cost of the energy wasted in the cable equal to the annual charge (interest and depreciation) on that portion of the cable's cost which varies with the cross-sectional area.

It will be noticed in the first place that the solution is independent of the length of the cable, every item of charge being directly proportional to length. Furthermore, in order to isolate the issue as much as possible, every item not varying with the area is omitted entirely—thus the working cost of the service is taken not as the whole energy input, but simply as that part of the input which does not come usefully out at the other end. In a similar way the cost of laying or fixing is assumed to be either independent of the area (as it probably is, within the range of choice in question), or else that this,

together with all other costs, such as insulation, insurance, right of way, repairs, etc., can be capitalised in the form $(\alpha A + \beta)$ £ per unit length of cable, where A is the cross-sectional area and α and β are constants.

The following symbols will be employed in the proof of this law :—

I = the line current (constant or full load value).
L = the total length.
A = the cross-sectional area of one conductor
ρ = the specific resistance
$(\alpha A + \beta)$ £ = the cost of unit length of cable

All in the same (*e.g.*, inch) units.

p = the price of energy in pence per kWh.
r = combined rate of interest plus depreciation (on unity).

The basis of comparison is to be annual costs or charges, and expressing both groups in the same units :—

Structural charge $S = L\,(\alpha A + \beta)\,r$ £ per annum.

$$\text{Working charge } W = \frac{I^2 L\rho}{A} \times \frac{24 \times 365}{1{,}000} \times \frac{p}{240} \text{ £ per annum}$$

(assuming that the current flows steadily throughout the year). Hence the total relevant charge for the service is given by :—

$$T = L\alpha A r + L\beta r + 0{\cdot}0365\,\frac{I^2 L\rho p}{A}.$$

The similarity of this with the general equation at the bottom of the chart on p. 95 will be obvious, and furthermore it will be evident that by differentiating the total with respect to A and equating $\frac{dT}{dA}$ to zero, the constant middle term will disappear, and the minimum cost will be obtained when the two variable charges are equal.* Thus :—

$\frac{dT}{dA} = L\alpha r - \frac{0{\cdot}0365 I^2 L\rho p}{A^2}$, which is zero when $L\alpha A r = \frac{0{\cdot}0365 I^2 L\rho p}{A}$, thus proving the law as stated above.

* In general, when the two charges are directly and inversely proportional to a variable (so that their product is constant), their sum is a minimum when they are equal. This can be shown not only by the calculus, but also graphically or by numbers, *e.g.*, with the pairs of numbers 16 and 1, 8 and 2, 4 and 4, all multiplying to 16, the smallest pair is the one in which the two numbers are equal.

It follows from this that $\frac{I^2}{A^2} = \frac{ar}{0{\cdot}0365\rho p}$, and putting in the value of ρ for 20° C. (0·677 microhms per inch cube), and putting in $a' = 36a$ (so that the cable price can be per yard instead of per inch), this gives, in English units, $\frac{I}{A} = 1060\sqrt{\frac{a'r}{p}}$ amperes per square inch.

It will be seen that the solution can be put in the form of a current density, and, in fact, Kelvin's law can be stated as "The most economical *current density* is that which makes the annual cost, etc., etc." In the author's opinion this form, although not the one usually adopted, is preferable for a number of reasons. In the first place, it gives more information than the other form, *i.e.*, a solution applicable to any value of the current. Secondly, it indicates immediately if the solution is inadmissible on account of heating considerations. Finally, it emphasises the fact that the structure is homogeneous, so that the problem of what section to employ for a given current is essentially the same as the problem of the most economical current for a cable already laid.

Application and Graphical Solution.—In order to illustrate the application of the above law, the following simple problem will be considered :—

Example 1.—Find the most economical section of single-core L.T. paper-insulated (armoured) cable to transmit (*a*) 120 amperes, and (*b*) 240 amperes continuously with energy at $\frac{1}{2}d.$ a unit and interest at 6 per cent. per annum, given the following price for the cable, which can be assumed to have a useful life of twenty years and no salvage value :—

C.S.A. (sq. in.) .	0·0225	0·04	0·075	0·12	0·25	0·50
Price per 1,000 yds. (£)	98	124	170	224	356	605

The first step is to discover the law followed by the cable price. If this can be assumed to be a linear one ($aA + \beta$) the values of the constants can easily be found by constructing two simultaneous equations, using any pair of prices. Failing this assumption, it is necessary to plot the figures, and when this is done it will be found that they lie very nearly on a straight line, the law of which is ($1060A + 83$) £ per 1,000 yards, so that a'

bove equation is 1·06. The end-of-year depreciation
iecessary to realise 100 in twenty years at 6 per cent.
ım (Appendix II.) is 2·72, so that the combined rate
72, and this gives the economical current density

$$\text{30}\sqrt{\frac{1\cdot06 \times 0\cdot0872}{p}} = \frac{322}{\sqrt{p}} = 455 \text{ amperes per square}$$

follows that the most economical section to employ in

ting 120 amperes will be $\frac{120}{455} = 0\cdot264$ square inches,

nearest available size to this is 0·25 square inches. In
ting 240 amperes, the most economical section will be
his, and the nearest available cable is 0·50 square

bove problem can be solved graphically, and this
has several advantages. The base of the graph will
t size of structure, measured in C.S.A., and the ordi-
ll show the annual costs or charges per 1,000 yards of
For this purpose the actual cable prices will be used,
ormula, and multiplying these by the combined rate of
plus depreciation (0·0872), the line "structural charge
g. 5) is obtained, which it will be noticed is very nearly
The only working charge which is to be considered
ue to the energy wasted in the cable, and since this is

$$\frac{I^2L\rho p}{A} \times 0\cdot0365 = \frac{6\cdot4}{A} \text{ £ per annum for 1,000 yards}$$

t will be inversely proportional to the C.S.A., so that
e will be a hyperbola (W).
g the two curves gives a total (T) which has its mini-
lue at 0·264 square inches. If the current to be trans-
vere 240 amperes, the W curve (shown dotted) would
r times its previous value, since losses are proportional
nt squared. Adding this hyperbola to the S curve
new total T (shown dotted) which is a minimum at
uare inches.
l be seen that the graphical method does not necessitate
te plotting of the cable prices, and since it makes no
ions as to the law followed by the cable cost, it offers a
neral solution than is afforded by Kelvin's law in its
gebraical form. At the same time it illustrates the truth
aw, since the position of the minimum total depends only
elative slopes of the two components and not on their

absolute values, and at the minimum point the wasted energy charge is equal to that portion of the standing charge which varies with A (see arrow lines). Another advantage of the graphical solution is that when the ideal section is not obtain-

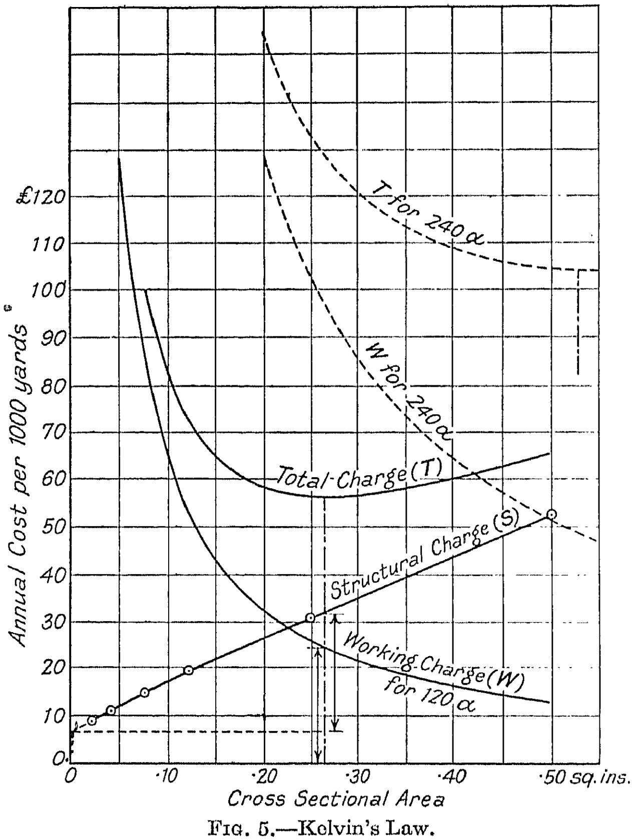


FIG. 5.—Kelvin's Law.

able the curve shows which of the available sections is the most economical.

Economy of Cable already Laid.—The type of problem which was referred to on p. 90, in which the structure is fixed rather than the service (a case which is hardly likely to occur in practice),* would arise in finding the most economical current

* Also, if such a case did arise it is probable that the value or earning capacity per ampere carried would not be a constant quantity, so that what follows here would not be applicable.

to transmit along a cable already laid. Defining economy as the ratio of the values of output to input, which in this case can be taken as $\frac{I}{S + W}$, where I is the current carried, it will be clear that maximum economy will be attained when the ratio $\frac{S + W}{I}$ is a minimum, *i.e.*, when $\frac{S}{I} + \frac{W}{I}$ is a minimum, *i.e.*, when the standing charge per ampere plus the wasted energy charge per ampere total a minimum. In solving graphically a suitable base scale would be current density $\frac{I}{A}$, and on this base the working charge per ampere $\frac{W}{I}$ would be a straight line, and the standing charge per ampere $\frac{S}{I}$ would be a hyperbola (omitting the constant portion of the cable cost). The minimum therefore occurs where the two are equal, which, of course, is at the current density already found; and this agreement again emphasises the fact that one is dealing with a homogeneous structure, so that the solution of the most economical current for a fixed cable is the same as that of the most economical section for a fixed current.

Truth of Law.—Amongst many engineers Kelvin's law is regarded with mistrust, and cable sizes are frequently chosen without any regard to the economic principle at all. The law itself is sometimes accused of being untrue, or if not untrue, at least inapplicable to particular cases, and it is important to consider the grounds for these charges. With regard to the truth of the law that least expense occurs when the two charges are equal, this must follow mathematically if the cable cost and the wasted energy cost (both expressed annually) are each linear functions of the C.S.A., the former being a direct and the latter an inverse function; and the law can only be untrue if one or other of these assumptions is untrue.

As regards the former, it will be found that in the majority of cases and over a wide range of sizes, the cable first costs do follow very closely a linear law, *i.e.*, $\alpha A + \beta$, where A is the C.S.A., and α and β are constants. The reason for this is that the materials cost of the copper alone frequently accounts for more than half the total price, and this is necessarily pro-

portional to C.S.A., whilst the remaining materials and the labour and overhead charges can easily be grouped into a proportional and a constant portion. Thus the element in the total cost which is not proportional to C.S.A. is small, so that even if this is not absolutely constant, but fluctuates somewhat with the area, the discrepancy is slight. In the case of the cable whose prices were employed in the above example, namely armoured single core paper insulated, the " S " curve in Fig. 5 shows that the price points are almost exactly on a straight line, and, furthermore, that the constant portion is a small one.

As further illustration of this point, the following figures may be cited from a recent quotation for unarmoured three-core cable of the paper insulated type. The figures were plotted exactly as received without any omissions, and in each case a linear relationship was established without difficulty, the maximum deviation of any of the figures from the straight lines being only $4\frac{1}{2}$ per cent.

Working Pressure.	Range of C.S.A. (A).	No. of Prices received.	Nearest Linear Price Formula (£ per 1000 yards).	Biggest deviation of actual Prices from this.
				Per cent.
660 V.	0·0225 — 0·50	13	4150A. + 90	4·2
6,600 V.	0·0225 — 0·25	10	4320A. + 200	3·6
33,000 V.	0·0225 — 0·25	6	3560A. + 1065	4·4

A striking point is that the proportional element is very nearly the same in the three cases, namely, about 4A. £ per yard, so that practically the whole of the extra cost of a high tension cable is the constant element, and the extra for a larger size is no greater than it would be on a medium or low tension cable. This has a very obvious bearing upon the question of economic choice of section, since for this purpose the constant portion does not enter into the problem at all.

Taking the price *per core* of the low tension cable, namely, (1383A. + 30) £ per 1,000 yards, it is interesting to compare this with the price of a single-core cable to the same specification. A quotation received at the same time as the above for single-core lead-covered unarmoured cable and covering the same range of sections showed a price per 1,000 yards of (1320A. + 50) £, none of the figures differing from this by more

than a few per cent., and again the similarity of the proportional element is very striking. Thus it appears that not only do cable prices follow very closely a straight line law, but in the case of the unarmoured paper cables mentioned above, the proportional element per core (which determines the economic size in any given case) is very nearly the same in all the four cases investigated, namely, about $1\frac{1}{3}$ A £ per yard (*i.e.*, 4 A £ for three-core).

The cost of laying, including such items as wayleaves, etc., can also be brought in, but in all probability this will not vary within the range of choice in question, and even if it does it can usually be fitted into the $\alpha A + \beta$ formula. Another possible discrepancy lies in the fact that although the first cost may be a linear function of the C.S.A., the annual cost may not be so, since a difference in section may mean a difference in the length of the working life, and so a change in the depreciation rate. This, however, is a purely academic point, which is hardly likely to merit consideration in practice.

As regards the assumption that the losses are inversely proportional to the C.S.A., this is true of the I^2R losses provided the variation of R with temperature can be neglected. In low tension D.C. cables these will be the only losses, and in almost all cases they will form the great majority. Other losses which might occur, such as the eddy current, dielectric and corona losses, are likely to be almost negligible in comparison, except with a very large unstranded conductor, and except when the frequency or pressure are abnormally high. It is therefore doubtful whether these departures from strict proportionality will be sufficient to upset the truth of Kelvin's law in any but exceptional cases.

Another running cost which might be affected by the choice of C.S.A. is the repairs and maintenance item, and in some cases it may be necessary to include this in the calculation. In conclusion, it will be noted that the *principle* of economic choice illustrated in Kelvin's law is just as applicable even if the variables are not linear functions. It is only necessary to plot the two sets of annual costs to a base of C.S.A. and add them to find at what point the total is a minimum. Thus in the graphical solution, Fig. 5, it would not have mattered if the "S" curve had not been straight or the "W" curve a hyperbola: provided the problem furnished sufficient data to plot them, their sum would always indicate the position of minimum total cost.

Applicability: Steady Currents.—With regard to the more serious charge levied against Kelvin's law, that it is inapplicable or irrelevant, this is a possibility with any purely economic determination. Kelvin's law concerns the balance which should exist between expenditure on structure (*i.e.*, cable size), and expenditure on working (*i.e.*, losses), and it was seen in Chapter V. (p. 86) that there are often three separate and distinct ways of determining this balance. In the case of a cable there is the economic limit, the heating limit, and the regulation (voltage drop) limit, any one of which may be used in fixing the cable size. It will be noted that the second and third represent definite limits or stops below which the plant size must not go, whereas the first merely gives a point of maximum economy on either side of which the cost will be greater than it need be. In the case of cables the first and second are independent of the length, whilst the third is directly proportional to length; the first is a matter of costs and the second of physical conditions, so that all three depend on quite different and unrelated factors. In every doubtful case all three sets of calculations should therefore be carried out, and the size chosen should be that indicated by the largest of the three results.

In what follows, the regulation limit will not be considered, since it does not lend itself to general treatment. Being a function of length, it must almost of necessity be considered separately for each particular case. As regards the other two, namely the economic and the heating limits, the important thing is to discover under what conditions the economic calculation is likely to indicate the larger section, and so be the determining factor. For if Kelvin's law is likely to indicate in the majority of cases a section too small on grounds of heating, its usefulness is clearly not very great.

When the current is a steady and continuous one throughout the twenty-four hours and each day of the year, there is no doubt that the economical current density is well within the limit allowable on heating grounds in all but exceptional cases. It has been seen that the most economical density is given (in English units) by $\frac{I}{A} = 1060\sqrt{\frac{a'r}{p}}$, and taking the single-core armoured cable on which an example has already been worked, with life and rate of interest as before $\frac{I}{A} = \frac{322}{\sqrt{p}}$, *i.e.*, the eco-

nomical current density in amperes per square inch equals 322 divided by the square root of the energy price (in pence). This latter should be the price to the authorities laying down the cable, since it is their costs which are to be a minimum; and when this figure is available as a single overall price per unit the law can easily be applied and will not give a density exceeding 1,500 amperes per square inch for this type of cable unless the energy price is one twenty-second of a penny or under.

Since only the losses in a single core have been considered in the formula, it follows that in a multi-core cable a' should refer to the price *per core, i.e.*, for the 660 V. unarmoured cable referred to on p. 109, $a' = \frac{4 \cdot 15}{3}$£ per yard. Taking the same life and rate of interest as before, the economical current density $\frac{I}{A}$ will then be given by $\frac{368}{\sqrt{p}}$, which again will not give a density exceeding 1,500 amperes per square inch unless the energy price is one-sixteenth of a penny or under. For the high tension 3-core cables very similar results will be obtained, since the proportional element of cost is substantially the same, and the exact figure for the economical density will be $\frac{376}{\sqrt{p}}$ for the 6,600 V cable, and $\frac{341}{\sqrt{p}}$ for the 33,000 V. It will therefore be seen that at present-day prices for paper insulated cables, the solution is well on the right side of the heating limit unless the cost of energy is exceptionally low.*

Application to Varying Currents.—When the current alternates, whilst keeping to a steady R.M.S. value I throughout the twenty-four hours, the solution is the same as the above, since the annual losses will still be represented by $I^2R \times 24 \times 365$, but when the effective current (direct or root mean square) fluctuates during the day, the position will be entirely different. Thus if the current I in the above formulæ is carried for, say, six hours every day and no current passes for the remainder of the day, the danger of the heating limit being reached will be just the same as if it flowed continuously, whereas the losses will be only a quarter, so that I^2/A^2 will be four times (see p. 104),

* The permissible density on heating grounds varies with the size and type of cable, the figure of 1,500 amperes per square inch being mentioned merely as a basis of comparison. In many cases the heating limit is higher than this, thus making the sphere of economic choice still wider.

i.e., the economical current density will be double. More generally, if F is the fraction of the day during which full load current flows (nothing flowing during the remainder of the day), the losses will be F times their previous value, and the economical current density $\left(\frac{I}{A}\right)$ in English units will equal $1060\sqrt{\frac{a'r}{Fp}}$. Thus when $F = \frac{1}{4}$ this becomes $2 \times 1060\sqrt{\frac{a'r}{p}}$, *i.e.*, twice the original value.

Instead of the full load current being carried for a quarter of the day there might be a current of approximately a quarter load carried for the whole of the day, the full load current being only reached occasionally during the year. Provided that on these occasions this peak current lasted long enough to bring the cable to its maximum temperature, it would still determine the heating limit, although the economic limit would be determined by the losses—in this case only one-sixteenth as much as before (since virtually the current is only a quarter). The economical density in the case cited above would now be given by $\frac{I}{A} = 1060\sqrt{\frac{a'r}{\frac{1}{16}p}} = 4 \times 1060\sqrt{\frac{a'r}{p}}$, *i.e.*, four times the original value.

Both the cases cited above would correspond to a load factor of 25 per cent., but the most economical cable sections would be respectively one-half and one-quarter of the original values, whilst the heating limit would be unaltered. In practice a load factor of this value is unlikely to represent either of these extreme cases, so that the economic solution will lie somewhere between the two results.

Summing up the above, it may be said that when the load on a cable fluctuates during the day, it is impossible to apply Kelvin's law unless the loading corresponds to one of the extremes mentioned above, or unless a mean load curve for the year is available; and in this latter case it is laborious to apply, since the root mean square value of the curve must first be worked out. When applied it will give a higher current density (and therefore one less likely to be acceptable on heating grounds) the smaller the consumption; and this will be particularly so when the consumption is in the form of a fairly uniform low loading with only occasional peaks. Taking the worst case which could arise on a cable whose mean annual

load factor was 25 per cent. (which would be when approximately one-quarter full load current flowed almost continuously), the economical density for the single-core armoured cable already instanced would be 1,500 amperes per square inch or over when the price of energy was $\frac{3}{4}d.$ or under.

Other Difficulties.—Whilst the difficulty concerned with fluctuating currents is the most serious one in limiting the useful applications of Kelvin's law, there are other difficulties, and other criticisms which have been levied, although some of these are due to misunderstandings of the law itself. It must be clearly realised that Kelvin's law is a rule for variations in A, all other items being constant, and it cannot be applied for I variable, as in a problem to find the most economical voltage. It assumes that the current transmitted and the voltage at the receiving end are fixed, these being involved in giving a definite service to the consumer; so that the price of energy in the formula is calculated from the cost (at the transmitting end) of the extra power which has to be applied in order to compensate for the voltage drop due to cable resistance. Any attempt to extend Kelvin's law to the economics of transmission as a whole (consumer's voltage not fixed) will be found to involve very extensive difficulties.

Another difficulty is that Kelvin's law implies one fixed price for the energy entering the cable, whereas actually the value or cost of this energy to the authority supplying it (who is assumed also to be the authority laying down the cable) will usually vary considerably at different times of the day and year. This does not affect the principle of the law which equates cost of losses to cost of cable copper, but it makes it much more difficult to apply. For the cost of the losses must now be the sum of all the terms of the form I^2Rtp, where t represents the time in hours, and p is the energy price during those hours.

Additional Worked Examples

2. In the case of the V.I.R. cable whose prices are given below, find the most economical section to transmit 20 amperes continuously with energy at 0·4*d*. per unit, and find for what energy price the economical density would be 1,500 amperes per square inch. Take interest at 6 per cent. with a life of twenty years, and salvage value zero.

Cable size .	7/·052	7/·064	19/·052	19/·064	19/·072
Area sq. in. .	0·0145	0·0225	0·040	0·060	0·075
Price per mile	£36 5*s*.	£47 16*s*.	£83 6*s*.	£118 1*s*.	£150 10*s*.

By plotting the cable prices, these are seen to lie very nearly on a straight line, the law of which is (1,860 A + 8) £ per mile. Also from Appendix II., $r = 0{\cdot}06 + 0{\cdot}0272 = 0{\cdot}0872$.

Hence economical density

$$\frac{I}{A} = 1{,}060\sqrt{\frac{a'r}{p}} = 1{,}060\sqrt{\frac{1860}{1760} \times \frac{0{\cdot}0872}{0{\cdot}4}} = 509 \text{ amperes per}$$

square inch, so that for a current of 20 amperes the most economical section = 0·0393 square inches, and the most suitable size is 19/·052 (0·040 square inches).

For an economical density of 1,500 amperes per square inch, the energy price would have to be $0{\cdot}4 \times \left(\frac{509}{1{,}500}\right)^2 = 0{\cdot}0456d.$ or one twenty-second of a penny.

3. A certain transmission system employs the single-core armoured paper insulated cable mentioned in the text, of which the first cost is given by (1,060 A + 83) £ per 1,000 yards. The maximum current in the year is 240 amperes, and the mean annual load factor is $33\frac{1}{3}$ per cent. Find the most economical current density at which to work (a) if the load consists of 240 amperes for eight hours each day, and (b) if the load consists of 80 amperes practically continuously (the maximum current being taken only occasionally in the year). Take energy at $\frac{1}{2}d$. a unit, interest at 6 per cent. per annum, twenty years' life and no salvage value.

As before, $r = 0{\cdot}0872$ and $a' = 1{\cdot}06$ per yard.

(a) The full current flows for a fraction F of the day where $F = \frac{1}{3}$. Hence economical density

$$\frac{I}{A} = 1{,}060\sqrt{\frac{ra'}{pF}} = 1{,}060\sqrt{\frac{0{\cdot}0872 \times 1{\cdot}06}{\frac{1}{2} \times \frac{1}{3}}} = 788 \text{ amperes per}$$

square inch, requiring a section as near as possible to 0·030 square inches for 240 amperes.

(b) If I still represents the maximum current in the year (240 amperes), the virtual current from the point of view of losses will be one-third of this, and I^2R will be one-ninth. Hence the economical density $\frac{I}{A} = 1{,}060\sqrt{\frac{0{\cdot}0872 \times 1{\cdot}06}{\frac{1}{2} \times \frac{1}{9}}} = 1{,}364$ amperes per square inch, requiring a section of 0·0176 square inches for 240 amperes.

Alternatively, this part of the question could be worked out for 80 amperes (flowing continuously), giving a density as found on p. 106 of 455 amperes per square inch. Hence section required $= \frac{80}{455} = 0{\cdot}0176$ square inches.

CHAPTER VII

MOTORS: PARTICULAR CASES

Paying for Efficiency.—It will be clear that there are a great many different economic problems that might arise in connection with the choice of a power unit. For instance, there might be alternative sources of supply available, so that either an A.C. or a D.C. motor can be installed. More usually the choice will relate to something less wide, *e.g.*, high speed with reduction gearing as against low speed, or low voltage motor plus transformer as against high voltage motor. Such problems clearly cover a very wide range, and there is little in the way of a general solution that can be given. All the various factors must, as far as possible, be assigned their cash values, and the outstanding balance can then be struck on economic lines by the application of the general principles given in the preceding chapters. In most cases, however, the physical limits and local conditions will determine these matters, and the only problems here dealt with concern the choice of a given type of motor to give a fixed power and speed for a given number of hours a year.

Cases for the economic choice of such a motor may arise in a variety of ways. The simplest is that in which two or more machines are available with identical or equivalent properties (so far as the purchaser is aware), one of which is cheaper than the others. Choice will then naturally fall on the cheaper one, but this is so obvious that it hardly calls for comment here; and the same thing would apply to a still greater extent if, instead of being identical, the cheaper motor were actually better. But where the more expensive machine possesses compensating qualities to which can be assigned a cash value, then a specifically economic choice may be made, since the question must be settled as to whether or not those advantages are worth the extra first cost demanded for them.

The superior qualities and advantages of the more expensive motor may consist of an anticipated longer life or freedom from trouble, but for the most part the only differences likely to arise and be a factor in such a choice are differences either in

efficiency or power factor. As a rule, the prospective purchaser will only obtain tenders from those he considers to be first-class makers ; amongst such there is not likely to be any appreciable difference as regards either life or reliability, and any such differences can usually be regarded as having been weeded out before the stage is reached in which economic choice is a factor. So far as the present book is concerned, only these two differences will be dealt with, efficiency being treated here, and power factor in the third section of the book. Sometimes the choice lies not between two or three specific alternatives, but over a whole range of possibilities embodying progressively better efficiencies at greater first costs. The question then is, given a certain tendency, say, in the direction of increased copper and iron sections, to what extent is it economic to pursue it in any given case ; and this question is considered in the chapter which follows.

What has already been said about the two ways of choosing the size or quality of electrical apparatus applies fully to the case of motors. The normal method is to select a machine having just sufficient iron, copper, etc., so that (when giving the output and occasional overloads required) the temperature-rise, centrifugal and dielectric stresses, the commutation and the regulation are all kept within satisfactory limits. This may be described as the *physical* choice, and it will be obvious that in no case can a machine be installed smaller than this size. But if a bigger or higher grade * machine would have a better efficiency at the required output, it might be more economical in the long run to install it for this reason, and such a decision would then be a specifically *economic* choice. As the two sets of criteria are quite unrelated, it is evident that both calculations should be made, and the machine chosen should be the larger of the two, whichever that is.

An electric motor may be compared in this respect with a leaky vessel such as a slightly porous balloon. If too big a load is put upon it, it will burst and be destroyed, but long before this point is reached considerable leakage is taking place. Hence, if the cost of the inflating material is great compared with that of the balloon material, it may be cheaper not to load it up to its physical limit.

A single instance will serve to show how considerable is the

* The word "better" is not used because what is meant is not any superior merit as to essential points such as reliability, but merely the employment of higher quality materials having lower losses or giving better space factor.

cost of this leakage in comparison with the cost of the plant itself. A 40-h.p. 1,000-r.p.m. 50-period squirrel-cage induction motor having an efficiency of 88·5 per cent. can be bought for about £70 at the present time. Such a machine, running for eight hours a day and 300 days a year with energy at a uniform price of 1*d*. per unit, in a useful life of twenty years consumes nearly £7,000 worth of energy—equivalent in cost to five new motors every year of its life. Moreover, whilst the bulk of this energy is transformed into useful work, the sum actually wasted in machine losses amounts every year to £38 15*s*.—more than half the price of the motor.

When it is realised that the motor only has to be paid for once, and its cost can if necessary be spread over the whole of its useful life period, any gain in efficiency unless obtained at a very exorbitant first cost, would seem likely to pay for itself many times over. Thus if the above efficiency could be improved even by so much as ½ of 1 per cent. the energy saved would be worth £2 a year, which could be capitalised over a twenty-year life as equivalent to the lump sum of £25—sufficient to pay for a very generous increase in the copper and iron sections.

The small amount of attention paid to efficiency in its economic aspect is due very considerably to the difficulty of calculating, in any particular case, what degree of efficiency improvement would be worth while, and (when calculated) the difficulty of requisitioning such a machine. These two difficulties have been attacked in this and the next chapter on the lines of establishing a general criterion of the economic degree of efficiency, visualising for this purpose the employment, not solely of special high-efficiency machines, but also in some cases of standard machines built for a greater output than that required.

Methods of Improvement.—Since this book is dealing largely with the economics of choice, the question of efficiency improvement must be considered not so much from the designer's as from the purchaser's point of view. At the same time, it must not be forgotten that the former is really the fundamental one; the purchaser comes later, and he has usually no legislative opportunities, but merely powers of veto or consent, since he can only select from amongst what is provided. Hence the problem should really be dealt with at an earlier stage; it is only an unfortunate necessity which

compels the buyer to consider machines as they are rather than as they ought to be.

Broadly speaking, there are two distinct ways of increasing the efficiency of a machine of any particular output, which may be called the qualitative way and the quantitative way. The former consists in employing the same size of frame, but using higher-grade steel and better insulation; thus giving lower iron losses, a higher space-factor and therefore more copper and less copper losses. The quantitative method of improvement consists in employing a larger frame and therefore more iron and copper, thus decreasing the flux and current densities and therefore the losses.

The latter method of efficiency improvement is less sure than the former, since the larger frame will involve slightly more frictional loss, which may nullify some of the gain. It has, however, two advantages, the first of which is that any individual purchaser, knowing his own requirements, but having no influence with the manufacturers, may practise it and select the machine size which is most suitable for his particular conditions, even when he is not able to obtain alternative efficiency quotations.

The second advantage is that it involves no departure from standards, either in quotation or manufacture, so that it is possible to make a general survey of the advantages of economic choice (ranging over a large number of possible alternatives) without any information beyond that supplied by the makers' price list. It must, however, be understood that the quantitative method usually puts the worst side of the case for economic choice; and that where this method indicates an advantage to be gained from under-running, there will usually be a much greater advantage in qualitative improvements wherever these are commercially practicable. A combination of both methods of efficiency improvement is often possible for a large-scale purchaser of a number of duplicate machines, who, if he estimates on the purely quantitative basis and finds that a bigger efficiency is economically sound, may then be able to commission minor qualitative variations which will yield an even better economic result.

Economic Basis and Method: Loss Ratio.—The economic calculations involved in the choice of motors follow very closely the skeleton outline at the end of Chapter V. The most suitable basis of comparison in almost all cases is that of

annual costs or "charges," and the method adopted is to omit entirely all items which are common to every alternative. The only items which will have to be dealt with will then consist of a capital or structural charge (S) on account of the plant, and a running or working charge (W) on account of the energy wasted. With regard to the first cost of the plant, it will be assumed that the control gear, erection and other items are the same in all the alternatives considered; so that when a larger or higher grade machine is installed instead of a smaller or poorer one, the only increase in cost is the annual interest, depreciation and other capital charges on the larger machine, and the only decrease is the lessened cost of the machine losses.

These two varying charges can be regarded as the dependent variables in the problem, one of which increases and the other decreases, with an increase in the independent variable (the frame size or quality). Graphically, the last mentioned forms the base on which two curves can be plotted, whose sum will indicate the position of minimum cost or economic structure. This base could therefore be scaled in frame dimensions, rated output, efficiency, or other suitable unit; and in one of the author's papers published by the *Institution of Electrical Engineers* * several different scales were experimented with. But as a basis for a general consideration of all types, machine first cost is found to be the most suitable unit, and this has therefore been employed in most of the examples here.

In what follows, the first cost is denoted by C and the combined rate for interest plus depreciation by r. (If the machine has a salvage value at the end of its useful life, r must be adjusted in the manner explained in a previous chapter; and if other items, such as insurance, are affected by the choice of frame size, these may also be included in r.) The total capital charges which are relevant to the choice in question are then represented by Cr, *i.e.*, this is the annual cost of owning this particular machine instead of any other.

The other item of cost which will vary in the problem is that of the energy consumed, or more particularly the cost of that portion of the energy which is wasted in machine losses. For the total energy taken can be split up into two parts, the one (equal to the output) which is converted directly into useful work, and the other which supplies the losses. This latter, or loss component, is the only one which need be considered, since the other is common to all the alternatives in question,

* *Journal I.E.E.*, 1924, Vol. 62, No. 335.

and it is evident that it is dependent in some inverse way upon the efficiency (η).

The best way of expressing this characteristic of the machine is by means of the ratio, losses/output, which will be referred to as the loss ratio. Since $\eta = \frac{\text{Output}}{\text{Input}} = \frac{\text{Output}}{\text{Output} + \text{losses}}$, it follows that $1/\eta = 1 + \frac{\text{Losses}}{\text{Output}}$, so that the loss ratio is found from the reciprocal of the efficiency minus one. In the calculations which follow, the losses will be in watts and the output in horse-power or kVA. (This makes the calculations more convenient, and avoids unduly low values and the appearance of a large number of cyphers in the rate-of-change expression.) The loss ratio, denoted by Q, will therefore be found from $(1/\eta - 1) \times 746$ in the case of the motors and $(1/\eta - 1) \times 1{,}000$ in the case of transformers (Chapter IX.).

The importance of the quantity Q lies in the fact that, for any structure having this efficiency, the losses at any particular load can be obtained by simply multiplying Q by the load. The cost of the energy usefully converted in the machine is thus omitted from the problem, and the cost of the energy wasted is given by $Q \times$ horse-power output $\times$ hours of service per annum $\times$ cost (£) of energy per kWh divided by 1,000.

For simplicity, only efficiency will be considered in this chapter, and it will be assumed either that there is no difference in the case of the A.C. machines as regards power factor, or else that there is no penalisation for lagging loads. If, however, there is a wattless charge as well as a charge for kilowatt-hours, this can easily be brought into the problem (see Chapter XIII.). It will, moreover, be assumed that the service required is in every respect known beforehand, and that the only problem is to decide which particular motor of a given type will prove the most economical.

Choice of Quality: A.C. Motor.—In the type of problem first to be considered, alternative quotations are presumed to be available from makers of equal repute for machines of the same nominal size, but of different efficiencies and at correspondingly different prices. These variations may arise owing to actual differences in the quality of the materials, *e.g.*, higher grade steel and insulation, the latter giving a better space factor, and therefore more copper. They may also arise, even when there

are no qualitative differences, through the employment of what is really a larger machine, having more copper and iron, and therefore lower densities and losses. This often happens in practice, even though the makers have identical standards as regards rating, since a particular output may exactly fill one maker's frame and come midway between two frames with another maker. The latter must therefore quote for the larger frame with correspondingly easier densities, but with a higher price. Thus the distinction between qualitative and quantitative methods of efficiency improvement is not always easy to maintain, but in the second type of problem to be considered larger sizes of machine *of the same maker* will be selected.

Example 1.*—As an illustration of the first type of problem the following may be cited: A three-phase squirrel-cage induction motor was required for the purpose of driving a small battery booster requiring approximately 10 h.p. at 1,400 r.p.m. A number of quotations were received, the full load efficiencies ranging from 82 to 85 per cent., but several were weeded out on account of being at a disadvantage as regards both price and efficiency together. The three survivors, ranking equally in the purchaser's estimation as regards reputation, are shown in the table below, under the headings (*a*), (*b*) and (*c*).

The motor was required for use about eight hours a day and 300 days a year, and it would be on practically full load all the time it was connected, the energy being paid for at a uniform figure of a penny a unit. So far as was known, the use of the machine in this particular capacity would be required indefinitely, but in order to assign some period to the useful life, it may be assumed that this would be limited to twenty years, and that the salvage value at the end of this would be zero. There is no physical reason why it should not last much longer, and, on the other hand, it might have to be replaced for some reason in a shorter time than this, but in the latter case the salvage value might be appreciable and would help to balance the shorter life in the original capacity.

Interest is reckoned at 6 per cent. throughout, and at this figure the correct depreciation rate is 2·72 per cent. (Appendix II.). Multiplying the first cost in each case by 8·72 per cent. for interest and depreciation will then give the annual

* Examples (1) and (3) are reprinted from *The Electrical Times* of February 3rd, 1927.

cost of owning the motor, which is shown in the table under the heading "capital charge." The watts lost when giving 10 h.p. output, calculated from $(1/\eta - 1) \times 10 \times 746$, are shown in the next line; and multiplying this by the annual hours of service (8×300), times the price of energy ($\frac{1}{12}$*s.*) and dividing by 1,000 will give the annual cost of the wasted energy in shillings, which is entitled "inefficiency charge." Adding these two charges in the three cases gives the total relevant annual cost, from which it will be seen that machine (*c*), having the highest first cost, is actually much the cheapest of the three when total costs are considered. Even if the energy cost only ½*d.* a unit and the motor were in use only four hours a day, so that the inefficiency charges were just a quarter of the figures shown, machine (*c*) would still prove to be slightly the cheapest installation.

A.C. MOTORS: 10 H.P. SERVICE

	Machine (*a*).	Machine (*b*).	Machine (*c*).
Full load efficiency	82 per cent.	84 per cent.	85 per cent.
First cost	£21 17*s.* 0*d.*	£23 0*s.* 0*d.*	£25 5*s.* 0*d.*
Capital charge in shillings	38·1	40·1	44·0
Watts lost when giving 10 h.p.	1,637	1,421	1,316
Inefficiency charge in shillings	327·5	284·2	263·3
Sum of relevant charges in shillings	365·6	324·3	307·3

It may be objected that the saving is not great, only 17*s.* per annum in the case of (*c*) over (*b*), but this is only on one small machine and actually represents nearly 40 per cent. of the annual cost of the motor itself. Moreover, from calculations made elsewhere there is reason to believe that a similar degree of saving is usually possible on other sizes of induction motor, and in a large works employing electric power extensively, the saving to be effected on these lines might reach a very considerable figure.

If machine (*c*) is compared with (*a*) for the service in question it will be seen that the saving in the energy bill is nearly one and a half times the whole cost of the motor (both reckoned on an annual basis). From this emerges the extraordinary fact that for the service proposed, if machine (*c*) were priced at its present figure and machine (*a*) were given away, *it would still be considerably cheaper to buy machine* (*c*).

General Solution: Service-Price.—It will be noticed that in most of the economic calculations the working cost depends upon the product of two variables, hours of service and cost of energy, and it is therefore convenient to have the name for this product. The author has coined the term "service-price" to indicate the total number of hours of service per annum multiplied by the price of energy (£*s*). Thus a service of eight hours for 300 days a year with energy at 1*d.*, or four hours a day with energy at 2*d.*, or twenty-four hours a day with energy at $\frac{1}{3}$*d.*, will be represented by a service-price of 10. Another definition of service-price is to say that it is the cost in £ per annum for each kilowatt connected for the hours in question.

The above example may then be summed up for any set of conditions by saying that for all values of the service-price above 2, machine (*c*) will be the cheapest. This is an unusually low figure, corresponding as it does to energy at $\frac{1}{18}$*d.* a unit with continuous operation, or at $\frac{1}{6}$*d.* with operation for eight hours a day throughout the year. For a service-price between 2 and $\frac{1}{2}$, machine (*b*) will be the cheapest, whilst for figures below $\frac{1}{2}$ machine (*a*) has it.

It will be seen that the composite quantity service-price has a usefulness in economic calculations analogous to that of ampere-turns in magnetic questions. In each case it is the product of the two components which is important, and by having a name and symbol for this product a much more general solution becomes possible. Service-price includes the two main varying elements, namely, the loading factor and the cost of supply; and the chief limitation to its use is when the hours of service are not all at full load, or when the cost of supply cannot be expressed as a simple price per unit. In all cases in which the value of the service-price is obtainable, the annual cost of the energy wasted (or inefficiency charge) will be given by Q times the service-price (£ per annum) when Q is the loss ratio.

Choice of Size.—The economic advantage of the more expensive machines in the above instance is so marked that one is forced to the conclusion that in many cases it would pay to enlarge a machine deliberately, beyond the size necessitated on physical grounds, merely in order to take advantage of the improved efficiency resulting from the lower copper and iron densities. This possibility is the one referred to above as the quantitative method of efficiency improvement, and a particular case is worked out below, and a general one, over a complete range of sizes, in the next chapter.

There is one great advantage about this method of efficiency improvement: it avoids accidental variations and differences between different makers, since the alternatives can then be confined entirely to one maker and type. Often the alternative machines of an ordinary quotation are put forward by firms which do not rank quite equally in the mind of the purchaser, and this prevents the economic choice from presenting itself in the clear-cut manner of the last example. Often there may be a suspicion that the slightly higher efficiency quoted by one firm represents merely a greater degree of optimism, and that if the two machines were put on a test-plate there might be nothing in it between them. A much more indisputable case would therefore present itself if any one maker, using the same type of construction in both cases, would quote an alternative machine with a higher efficiency at an appropriately higher price.

Unfortunately it is rarely that a maker can be persuaded to do this, at least for an individual small-scale buyer, as it will usually mean some departure from standard, and this is itself uneconomical and likely to increase the price by a disproportionate amount. This is particularly true of the large maker; and probably the simplest way, under present manufacturing conditions, of obtaining a better efficiency from the motors of any one firm, is by the employment of an absolutely standard machine of a larger size than is physically necessary, and under-running it. Such a suggestion affords the purchaser, even of a single machine, the opportunity of exploring the possibility of economic choice; this being based on the particular conditions of service, price of energy, etc., for which the motor is required, and with which probably no one but the purchaser is fully acquainted.

The possibility of economic choice of size clearly rests on two propositions: that a large machine has a bigger efficiency

than a small one, and that this bigger efficiency can be maintained even when the machine is under-run so as to give the smaller output. These two propositions are both discussed at the commencement of the next chapter, and the second one is dealt with in detail in Appendix IV. For the purposes of the present chapter it is not necessary to assume that the bigger efficiency of the larger machine is *fully* maintained on under-running, but merely that it remains substantially greater than that of the smaller machine. Provided that the under-running is not excessive, and that it is carried out in the manner laid down in the Appendix, this will almost always be the case.

Choice of Size: Small D.C. Motor.—Since the larger the machine the greater its efficiency, and since, on the other hand, the efficiency can never be greater than 100 per cent., it follows that the larger the machine the more nearly does its efficiency approach unity, and the less room there is for further improvement. It would therefore appear upon a superficial view that the most promising field for the application of economic methods of choice would lie among small sizes, although it must not be forgotten that the value of a given efficiency improvement is larger the greater the output.

Another advantage which undoubtedly accrues in the case of small machines is that the individual user may be a large-scale purchaser, employing the machines for ventilation, loom driving, etc., and may be in a position to commission qualitative alterations if the calculations indicate that this is likely to be advantageous. In the following example a fractional horse-power D.C. motor service is considered, and it will be found that the saving through the use of a large machine, although small in itself, is quite big in proportion to the other costs involved.

Example 2.*—The case to be considered here is one in which the service required is $\frac{1}{8}$ h.p. at 1,000 r.p.m. from a shunt motor on D.C. mains. A number of quotations were obtained for motors of this speed ranging from $\frac{1}{8}$ h.p. to 1 h.p., and one set of these, as shown in the first four columns of the table, was selected. (All of these figures were taken from a single quotation, with the exception of frame *b*, which was taken from a precisely similar range of machines by another maker. In all

* This example is taken from the author's paper, *Journal I.E.E.*, 1924, Vol. 62, No. 335.

cases except the first, the prices include an extra for supplying to a non-standard voltage. The selection of a quotation was made as far as possible with a view to getting a typical as well as a sufficiently continuous range of figures, but from a comparison with figures given later it would appear that the efficiencies of the ¼-h.p. to 1-h.p. sizes are slightly above the average usually obtained.)

The position then is that a certain service is required, for which machine (*a*), costing £5 16*s*. and having an efficiency of 53·5 per cent., is physically adequate. But by spending more money on the machine (a tendency represented by frames *b* to *e*) higher efficiencies can be obtained, and the problem is to find out to what extent (if any) this tendency should be followed. The first step is to estimate the probable lives and efficiencies of the larger machines when performing the ⅛ h.p. service.

With regard to lives, that of machine (*a*) has been taken as ten years, but it is reasonable to suppose that the lives of the larger machines will be progressively greater, both intrinsically and because they are giving so much less than their rated output. These lives are therefore taken as ranging from eleven to sixteen years, as shown in column 5, and taking interest at 5 per cent. and salvage value zero, the figures in column 6 are obtained by multiplying the first cost by the sum of the rates for interest and depreciation (Appendix II.).

D.C. Motors : ⅛ h.p. Service

Column 1.	Column 2.	Column 3.	Column 4.			Column 5.	Column 6.	Column 7.	Column 8.
Frame.	Rated Load.	Full-load Efficiency.	Price.			Estimated Life, *L*.	Capital Charge.	Estimated Efficiency when giving ⅛ h.p.	Corresponding Losses.
	H.p.	Per cent.	£	*s.*	*d.*	Years.	£	Per cent.	Watts.
a	⅛	53·5	5	16	0	10	0·750	53·5	81
b	⅕	63	6	12	0	11	0·800	59·6	63
c	¼	67	7	15	6	12	0·877	62·4	56
d	½	72	11	15	0	14	1·187	60·2	61·5
e	1	74·5	18	18	0	16	1·740	—	—

With regard to the efficiencies of the larger machines when under-run, and assuming suitable selection as regards voltage, etc., reasons are given elsewhere (p. 135 and Appendix IV.) for supposing that in the majority of cases, particularly if the

normal flux density is high or if the range of selection is not limited entirely by the printed price list, the full-load efficiency can be maintained almost intact down to one-half or less of the output. On the other hand, in a small machine, particularly if run at a high speed, the friction and other "constant" losses (not reducible by pressure or current reductions) may constitute a considerable proportion of the whole; and in order to make a conservative estimate, and also to illustrate the method as fully as possible, it will be assumed that in this case the irreducible losses amount to one-sixth of the total full-load loss. Column 7 shows the estimated efficiency (when giving $\frac{1}{8}$ h.p.) calculated on this basis, and it will be noticed that this rules out frames *d* and *e*.

The losses at $\frac{1}{8}$ h.p. output, corresponding to these efficiencies, are shown in column 8, and these figures are used as the abscissæ in Fig. 6. The base of this graph represents, in general terms, the frame size, but the actual scale employed is that of watts saved, so that the values [(1/efficiency) − 1] × 746/8 are scaled off from the right-hand side. On the above basis the upright lines, a, b, c and d, represent the first four frames quoted, and marking off ordinates to represent the corresponding structural or capital charges, the curve S is obtained. (Had the full-load efficiencies been maintainable for all degrees of under-running, the upright lines representing frames *b* to *d* would occur more to the right, and the chain-dotted curve S′ would have been obtained, frame *d* being now a useful contribution.) Curve S therefore represents the annual cost of saving energy by means of larger plant equipment; it rises slowly at first, but more steeply later, when the gain in efficiency for additional expenditure becomes steadily less.

With the base scale chosen it will be clear that curve W, representing the annual cost of the energy wasted in the motor, will be a straight line depending upon the annual hours of service and the price of energy. It has been seen that for a "service-price" of 10 (*e.g.*, eight hours × 300 days with energy at 1*d*. per unit) the working charge is £10 per kilowatt of inefficiency, and this gives a line W at 45 degrees to the axes. Adding curves S and W gives a total curve T *, which shows a minimum somewhere between frames *b* and *c*.

For any other service-price the line W will have a different inclination, and, in order to avoid redrawing, it can be reversed and made tangential to curve S at some point which will then

* For convenience this curve has been plotted minus £0·2.

indicate the economic position. Thus energy at half the above price, or hours of service proportionally shorter, would be represented by the reversed curve W′ which is tangential to curve S exactly at frame size *b*. On the other hand, a bigger

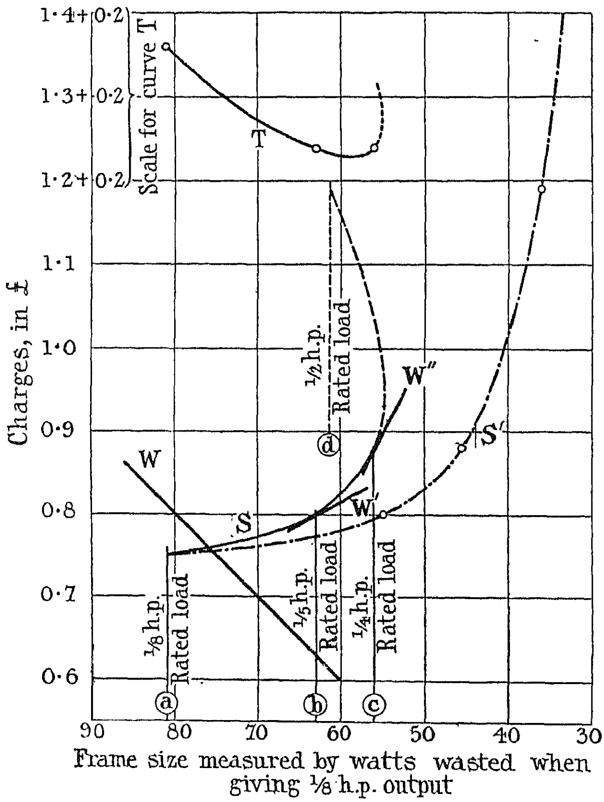


FIG. 6.—Fractional Horse-powers (D.C. Motors).

service price, *e.g.*, 20, would be represented by curve W″ which is tangential at about frame size *c*.

The economic advantages of under-running are here very apparent. Assuming a normal working day it will be seen that even with energy at only ½*d*. per unit it will pay to employ the next larger size of machine, whilst with dearer energy a still larger frame (or better still the same frame re-designed) is called for. The slope of curve S at its commencement is less

than half that of W′, showing that frame *a* is only economical for its rated output when energy costs less than $\frac{1}{4}d.$ per unit.

Choice of Size: A.C. Motor.—*Example* 3. As in the last case, in order to make a conservative estimate of the advantages of economic selection it will be assumed in the problem considered below that the machines are designed for maximum efficiency at full load, so that any degree of under-running will lower the efficiency somewhat. The service required is 40 h.p.

A.C. Motors: 40 h.p. Service

	Machine (*a*).	Machine (*b*).	Machine (*c*).
Rated at (h.p.) . .	40	50	60
First cost . . .	£69 18*s*. 0*d*.	£81 5*s*. 0*d*.	£92 0*s*. 0*d*.
Full Load Efficiency .	88·5 per cent.	89 per cent.	90 per cent.
Estimated efficiency when giving 40 h.p. .	88·5 per cent.	88·8 per cent.	89·4 per cent.
Watts lost when giving 40 h.p. . . .	3,875	3,760	3,540
Inefficiency charge (£s) .	38·75	37·60	35·40
Capital charge (£s) .	6·10	7·09	8·02
Sum of relevant charges	44·85	44·69	43·42

at 960 r.p.m., and it will be assumed that the life of the machine, the hours of service and price of energy are as in the previous A.C. example (twenty years, zero salvage value, 6 per cent. interest, service-price 10). The full load ratings shown in the table are taken from the standard price list of the maker selected, and cover machines of the same type, differing only in size, price and efficiency. From the last named can be calculated the efficiency when giving 40 h.p. output, and this is shown in the next line of the table. (The assumption made is that the full-load copper, iron and frictional losses are in the ratio 3 : 2 : 1, and that the larger machines are under-run in both pressure and current directions, the proportions being calculated from the formula given in Appendix IV.)

Calculating as before the annual costs of motor and wasted

energy respectively, and adding these it will be found that the largest size machine (*c*), even though its efficiency is appreciably reduced by under-running, is still the cheapest one to install, and shows a saving of £1 9*s*. per annum over machine (*a*). Had the design been slightly different, so that the full-load efficiencies were maintained on under-running, the annual saving would have been £3 12*s*., which is a quite appreciable item in view of the fact that the annual cost even of the largest machine tabled is only £8.

Summarising the results, it may be said that for any service-price higher than 4·2 (corresponding to energy at 0·4*d*. per unit for service on full load for eight hours a day and 300 days a year) machine (*c*) will be the cheapest, and for any lower service-price machine (*a*). The intermediate machine (*b*) has no advantage in either direction in this case, since whenever an improved efficiency is worth paying for, machine (*c*) will be found more worth its extra cost than is machine (*b*).

Summarising the results of the chapter as a whole, it will be seen that there is ample scope for utilising improved efficiencies, whether put forward by the makers or obtained by using larger machines, in all cases in which energy is not particularly cheap or the hours of full-load service unusually short. One difficulty which is experienced when the circumstances are particularly favourable to economic selection is to know when to stop; having found a certain tendency to be economically sound, the question naturally arises whether it may not usefully be pushed even further. When only certain definite alternatives are presented the difficulty does not arise, but from the point of view of the designer or large-scale purchaser some method is desirable of determining the whereabouts of the most economic point in a range of tendencies. This is attempted in the next chapter.

Additional Worked Example

4. A certain type of small fan motor can be built with cobalt steel permanent magnet fields at an extra (selling) price of 25*s*. per machine, and the field losses thus saved are 60 watts. If the useful life is assumed to be fifteen years, and the salvage value zero, find the service-price above which it would pay to install permanent magnet motors. Take interest at 6 per cent. and assume that all other items—insurance, maintenance, etc.—are unaffected.*

At the point where the two alternatives balance, the annual cost of the extra energy wasted in the less efficient motor (= kW × service-price) must equal the extra capital charge on the more efficient one (= extra first cost × rate for interest + depreciation)—both expressed in £ per annum.

$$\text{Hence } \frac{60}{1{,}000} \times S.P. = \frac{25}{20}(0{\cdot}06 + 0{\cdot}043) \text{ [Appendix II.].}$$

$$\text{Or } S.P. = \frac{100}{6} \times 1{\cdot}25 \times 0{\cdot}103 = 2{\cdot}1.$$

With continuous operation this would mean energy at only one-seventeenth of a penny.

* Some interesting data regarding this type of problem is to be found in a paper, "The Economic Aspect of the Utilisation of Permanent Magnets in Electrical Apparatus," by A. E. Watson, *Journal I.E.E.*, 1925, Vol. 63, p. 822 *et seq.*

Economics

CHAPTER VIII

MOTORS: GENERAL *

Effect of Under-Running.—The purpose of the present chapter is to extend the problem just considered, of the choice made by the individual purchaser between the two or three alternatives presented to him, to cover a much wider choice over a whole range of alternatives, and for any possible combination of power output, hours of service, and cost of energy. The previous chapter has shown that an advantage can often be gained by installing more or better active materials than are physically necessary; and the present chapter attempts to show how far this tendency should be pushed in any given situation, and at what point maximum economy occurs. The chapter is therefore intended not so much to assist the individual purchaser in a particular situation, but rather to explore the general possibilities of economic design and choice for all machines of the type in question. The possibility may not be immediately open to the small-scale buyer, and the solution may not coincide with any particular existing machine. It is, however, a question which will have to be faced by the designer of the future, once the present obsession with first cost and physical limits has passed away.

It has been seen that the best way of increasing the efficiency in any given case is by qualitative improvement, using the same sized frame, but for the purpose of a general treatment this method is set aside, and only the quantitative method—by under-running a larger frame—is here considered. It may be thought that, on any considerable scale, this is impracticable, but in the author's opinion there is no valid reason why a machine should not be used down to one-half or even a quarter of its rated output to meet exceptional circumstances. Provided the under-running is suitably proportioned, the efficiency need not come down to anything like the extent that might be feared from the shape of the ordinary efficiency curve, nor need

* The substance of this chapter is taken from the author's papers, *Journal I.E.E.*, Vols. 62 and 64, Nos. 335 and 351.

the commutation, power factor, regulation or other details of the performance be materially affected.

When a motor designed for maximum efficiency at full load or thereabouts is run at a lower load than this, the efficiency falls not because it is necessarily less for the smaller input, but because the balance between the iron and copper losses has been upset. To find its true efficiency at a lower load it is necessary to re-design it, or else to under-run it with regard to pressure simultaneously with the current reduction. Under these circumstances a given frame will give only slightly lower efficiency at a half or a quarter of its rated load than it does at full load.

In order to make this point clearer, suppose that a 1-h.p. shunt motor is to be run at about one-quarter of its load, and assume in the first place that the permeability of its field system is constant. If a 200-volt motor is selected and run at 100 volts, and loaded so as to take half full-load current, this means that the applied and back E.M.F.'s are both half their previous values, and, as the field is half what it was, this halved back E.M.F. will be produced by the same armature speed. In other words the speed will be the same as before. With the current and field each halved, the torque will be reduced to a quarter of its original value and (at the same speed) the power output will also be reduced in the same proportion. The armature and field copper losses will be one-quarter of their previous values, as will also the losses proportional to B^2.

Hence even with this extreme degree of under-running it will be only the friction, windage and a small proportion of the iron losses which will prevent the efficiency remaining constant and will cause the curve to tilt downwards slightly on the lower inputs. Furthermore, as the permeability will actually not be constant, but will be considerably better at the lower exciting voltage, the speed on half voltage will be less than normal and will have to be brought up either by means of the field resistance or by fitting a different set of field coils. Even in the former case the field loss will then be less than one-quarter of the full-load value, and this will still further tend to keep up the efficiency. (This question is fully dealt with in Appendix IV.)

As regards the commutation of such under-run machines, although it is true that the field is very much weakened, this will be found to be fully balanced by the reduction in the

armature current. The freedom from distortion of the field flux depends upon the *ratio* of the field and armature ampere-turns, and in fact, with each of these reduced to one-half, a slightly "stiffer" field can be anticipated, since with better permeability the tendency of the flux to spray will be diminished. Moreover, although the commutating E.M.F. for a given brush position will be halved, the armature current requiring reversal will also be halved. In such tests as the author has been able to make, the efficiency has been maintained quite as much as would be anticipated from the above reasoning, and in no case was the commutation noticeably inferior, even down to one-quarter of the rated output.

It must not be supposed that merely to select a large standard machine and under-run it on the above lines is in any way to be recommended as ideal, since in all cases some degree of re-design or rearrangement is preferable. All that is claimed is that such a course is quite practicable, even down to one-quarter of the rated output (at which point the running is liable to become unstable), whilst for small changes it may well prove to be the most economical way of increasing the iron and copper sections. In any case it provides a convenient hypothetical course of action, by means of which the economic advantages of plant increases can be tested.

Bearing this in mind it will be assumed that in all the cases considered and within the range of under-running proposed (usually not to a greater extent than down to 75 or 50 per cent. of the rated output) the full-load efficiency can be maintained intact. It is shown in the Appendix that this is true of static machines and of machines having small friction or large iron or copper losses, provided the under-running is suitably proportioned. It is not true of other machines, but as the chapter is intended to explore the general possibility of economic choice, it can generally be assumed that (especially where qualitative improvement is possible) an effect at least as good as that implied in the above assumption can always be obtained for the same or less cost.

Physical Rating of Electrical Machines.—It was pointed out in the previous chapter that the possibility of economic choice of size, *i.e.*, paying a greater first cost for bigger size in order to save running costs by better efficiency, rests on two propositions: that a large machine has a bigger efficiency than a

small one, and that this bigger efficiency can be maintained even when the machine is under-run so as to give the smaller output. The second of these propositions has been discussed in the foregoing paragraphs, and it is more fully treated in the Appendix, but the first propositional also requires some elucidation. One is apt to take for granted a better efficiency in a larger machine of any type, but in order to understand exactly why and how much, it is necessary first to consider what constitutes the normal or physical rating of a machine. A precise meaning can then be given to the terms "larger," "smaller," and "under-run," and their relationship to the efficiency of the machine will be made clear. It will be noted that the following remarks are intended to refer generally to any electrical machine, but in this book and for purposes of economic choice they are particularly applied to motors and transformers.

The physical rating of any given frame may be defined as that output which can be taken, for the period and under the conditions specified, without the temperature rise exceeding some figure which is considered safe from the point of view either of internal deterioration or of external danger. In addition, there is frequently an overload requirement for a short period, which may be a question of commutation or electric strength as well as temperature, and there is sometimes a requirement as to regulation, but for the most part it may be said that size is roughly determined in the great majority of cases by temperature limits.

It will be noticed that not one of the above limits concerns either the current or the flux densities employed, and yet the surprising thing is that over a wide range of sizes of any particular type of machine, a given temperature rise corresponds fairly closely to a given set of current and flux densities. This is one of those coincidences which are normally taken for granted, but which are not nearly so inevitable as they seem.

The close correspondence which exists between densities and temperature rises is illustrated by the fact that a d^2l formula is frequently used to estimate the frame size of a rotating machine to comply with a temperature specification, although the formula that output is proportional to d^2ln is based on the assumption of fixed densities and has no connection with temperature rise. In order to discover why the two largely go together, it will be well to take actual cases, bearing in mind

that with given densities the losses are each proportional to the volumes of the active materials.*

Taking as a starting point the assumption of fixed current and flux densities, consider first a constant-speed rotating machine. It is easy to show that if the specific electric loading is fixed, the output of such a machine is proportional to the d^2l of the armature or rotor, so that if d and l are both doubled the output will be increased eight times. On the other hand, the amount of iron in the magnetic path and the copper in the slots and end connections will only be slightly more than four times what they were, and hence these losses † will only be increased in this smaller ratio. As the rotating surface is just four times what it was and the peripheral speed is greater, the machine will thus be able to dissipate these losses with approximately the same temperature rise.

Hence the correspondence between densities and temperature rises is in a sense a happy accident ‡ due to the fact that the output (with fixed densities, speed, and dimensional proportions) is proportional to rotor volume (or d^3), whilst the losses and cooling capacity are both proportional to surface (or d^2).

Efficiency and Size.—Two things result from the above. Not only can designs be carried out on a basis of fixed current loadings and flux densities for a wide range of machines complying with a single temperature specification, but it also follows that the efficiency of a large machine must always be inherently better than that of a small one, and by a definite and ascertainable amount. For if the output is proportional to d^3 and the

* With constant frequency and flux density the iron losses can always be expressed per cubic centimetre of iron, whilst the copper loss $= I^2R = \frac{I^2l\rho}{A} = \left(\frac{I}{A}\right)^2 lA\rho$, so that this loss is also proportional to volume (lA) when the current density (I/A) is constant.

† The friction and windage losses are more difficult to estimate. The former will increase less than [illegible] with the increase in weight, whilst the latter will increase [illegible] the surface and because of the greater peripheral speed. These losses will therefore also be slightly more than four times what they were, but it will be noted that in any case they dissipate themselves with little or no effect upon the winding temperature rise.

‡ If the word "accident" be objected to, the position may be put the other way round by saying that in the design of a rotating machine the active materials are grouped round a cylinder to [illegible] or density, so that the losses shall always keep [illegible] the cylinder.

losses to d^2, the ratio (losses/output) will be proportional to $d^2/d^3 = 1/d = 1/\sqrt[3]{\text{Output}}$.

The extent to which this is borne out in practice can easily be discovered from any range of machines of the same type. Thus, on p. 143, a range of shunt motors from $\frac{1}{4}$ h.p. to 15 h.p. is listed, together with their efficiencies, from which can be calculated the ratio (losses/output), since this is equal to the reciprocal of the efficiency, minus one. Multiplying this figure by the cube root of the output (h.p.) gives a fairly uniform figure of approximately 0·4 in each case (see column 9 of table). Not only is there no perceptible tendency for this figure to vary with the machine size, but also the range of fluctuation is remarkably small, *i.e.*, 25 per cent. for a change in size of sixty times. Similar figures are listed for a range of squirrel-cage motors (column 8 of the table on p. 148), and a similar constant connection is obtained, although in this case there seems a tendency for the figure to increase with the frame size, thus suggesting that the root index should be somewhat higher than 3, due possibly to an increase in the specific electric loading.

In the case of transformers, if a uniform winding depth or loading were maintained as in rotating machines, the same law would hold, and the output would be proportional to the volume, or d^2l, of the iron core. But as it is customary also to increase the winding depth proportionally with the increase in the core (so using the whole of the " window " space) the output will go up at a greater rate than this.

By considering a simple transformer such as the ring type, it is easy to see that if d is the core diameter and if all the dimensions are varied in the same proportions, the iron cross-sectional area will be proportional to d^2 and the copper winding space to d^2, so that the output will be proportional to d^4, whilst the weight of both copper and iron, and therefore the losses, will be proportional to d^3. Hence the ratio losses/output ($= 1/\eta - 1$) is proportional to $d^3/d^4 = 1/d = 1/\sqrt[4]{\text{Output}}$. In a similar way it has been proved for all the commercial shapes of transformer, both shell and core types, that if all the dimensions are increased in the same ratio to a times their previous value, the output will be a^4 times and the loss ratio will be $1/a$ times its original value.*

* A. P. M. Fleming and K. M. Faye-Hansen: *Journal I.E.E.*, 1909, Vol. 42, p. 396.

It is therefore evident that if the heat-dissipating surface were only that of the transformer iron and windings, the temperature rise would tend to get steadily worse as the size increased, since losses are proportional to d^3, whilst surface is only proportional to d^2. This is what actually happens in the case of air-cooled transformers, and explains the virtual upper limit of size unless the densities are correspondingly reduced. In the case of oil-immersed transformers the radiating surface is that of the whole tank, and it is comparatively easy to proportion the size of tank and amount of oil either to the kVA capacity or to the actual watts lost.

The above statements must be modified slightly on account of change in the space factor. Thus the effect of doubling d will be to quadruple the winding space and to allow four times as many turns of the same size of wire. If, however, the current is to be increased rather than the pressure, a larger section of wire will be employed, giving a better space factor; so that the ampere-turns will be increased in the ratio of, say, d^3 rather than d^2. This will give an output of $d^2 \times d^3 = d^5$ times, an iron loss of d^3 times, and a copper loss of d^4 times. Hence the total losses (which can, of course, be distributed as desired) will lie between d^3 and d^4, say $d^{3\frac{1}{2}}$, giving a loss ratio proportional to $d^{3\frac{1}{2}}/d^5 = 1/d^{1\frac{1}{2}} = 1/(\text{Output})^{0\cdot3}$.

In confirmation of the above, the values of $(1/\eta - 1)$ for the single-phase transformers listed on p. 161 were plotted logarithmically to a base of rated kVA, and gave an index of 0·31, and the amount of individual deviation can be seen from the last column of the table. A similar line of three-phase transformers was also plotted and showed an index lying between $\frac{1}{3}$ and $\frac{1}{4}$; and, moreover, in each case the index showed a tendency to approach $\frac{1}{4}$ in the larger sizes, where the change in space factor would naturally be less marked. A further confirmation of the above reasoning was obtained in the case of the single-phase series by plotting (logarithmically) the iron cross-sectional area against the kVA output, which gave an index of 0·4, *i.e.*, the iron cross-sectional area is proportional to $(\text{output})^{0\cdot4} = (d^5)^{0\cdot4} = d^2$, as it should be.

It is needless to point out that the generalised reasoning and the few examples here given are in no sense put forward as an exact proof of the law followed, and in fact the precise form of this law is immaterial to the main object of this and the next chapter. What is important is to note that the efficiency of the larger machine is inherently better and by a definite

predicable amount, this amount being roughly given both for rotating machines and transformers in the statement that the loss ratio is inversely proportional to the output raised to a power lying usually between $\frac{1}{3}$ and $\frac{1}{4}$.

Another criticism which may be raised is that the process of explaining the index figure relating to transformers would appear to upset the figure arrived at for rotating machines, since in these also the space factor should improve with size. The two cases, however, are not parallel. In transformers the whole of the available window space is utilised, so that the space factor becomes a determining factor in the output obtainable from a given frame, whilst in a rotating machine a poor space factor can always be compensated for by increasing the slot depth : the limit here is not the space but the cooling capacity, and therefore the electric loading.

D.C. Motors : Graphical Treatment.—For the purpose of exploring the general possibilities of economic choice over a range of small D.C. motors, it has been thought best to take an average set of figures rather than one individual quotation. The type of machine to be considered is that of a shunt protected D.C. motor running at 1,000 r.p.m., and in columns 2 and 5 of the table below will be found the average of some half a dozen makers' net prices and full-load efficiencies for machines having rated outputs from $\frac{1}{4}$ h.p. to 15 h.p.

The base of the curves to be drawn will represent (as before) the size of the structure, but the scale in this case is in units of annual capital costs (interest plus depreciation), so that the "structural charges" curve S will be a straight line. The lives have been taken as ranging from twelve to twenty years (column 3), with a salvage value of 10 per cent. of the first cost in each case ; and the figures in column 4 are then obtained by using the table given in Appendix II. These are used along the base scale (Fig. 7) to erect thirteen upright lines representing the thirteen frame sizes, having the maximum output ratings shown and costing each year in interest and depreciation the amounts scaled along the base. As the ordinate scale in £ is the same as the base scale, a line S at 45° will represent the annual expense on account of structure.

As explained above, it will be assumed that for the degree of under-running proposed the full-load efficiency can be main-

tained intact by a suitable choice of voltage, etc.* Marking off these full-load efficiencies on the thirteen upright lines and drawing a smooth curve through the points, gives the curve shown at the bottom of Fig. 7; and it will be seen that (provided the makers' ratings fit in) it is possible, for an annual expenditure on hire and replacement of, say, £2, to purchase a frame having an efficiency of just over 78 per cent. The heating and sparking limit of this frame will prevent it being used above about $2\frac{1}{2}$ h.p., but for the moment it will be convenient to neglect temperature limits entirely and proceed as though any frame could be used for any output whatever.

Above the efficiency curve and to the same base has been plotted its reciprocal, the loss curve W, showing how the loss for any given output decreases with an increase in the size of frame employed. This curve will represent to some suitable vertical scale the annual cost of supplying the wasted energy, but the scale will depend not only upon the output, but also upon the hours of service and price of energy. Thus each particular horse-power and set of service conditions will have a particular W curve of the shape shown.

Reference has already been made to the term "service-price" to indicate the annual hours of service multiplied by price of energy (£), and it here becomes necessary to group with this the power output, and to use some term to indicate the product of the horse-power multiplied by the service-price. Since the power wasted in a motor equals Q times the output, when Q is the loss ratio ($[1/\eta - 1] \times 746$), it follows that the annual cost of this in £ will be $(Q/1{,}000) \times$ H.P. output $\times$ Hours of service $\times$ Price of Energy. Moreover, the last three items are constant relative to the frame size, and are mutually replaceable in the sense that, for example, 40 h.p. for eight hours a day is just the same economically as 80 h.p. for four hours. It is therefore well to have a composite term for the product of these three, and calling this product

* It is, of course, true that if frictional losses were a substantial item many of the actual sizes shown in the table would have an appreciably poorer efficiency if run at the next smaller listed rating. But the particular sizes listed must be regarded primarily as landmarks or points on a curve showing the trend of the manufacturing cost of improving the efficiency. In addition, since intermediate sizes are made, the individual user will frequently be able profitably to employ one of these (with a smaller degree of under-running), particularly if he can also command small changes in design such as new field coils, etc.

SMALL D.C. MOTORS.

Column 1. Rated Load.	Column 2. First Cost, C.		Column 3. Life.	Column 4. Corresponding Capital Charge.	Column 5. Rated Full-load Efficiency, η.	Column 6. Loss Ratio, Q.	Column 7. $-\frac{dQ}{dC}$	Column 8. Economical Service-price.	Column 9. $\left(\frac{1}{\eta}-1\right) \times \sqrt[3]{\text{h.p.}}$
H.p.	£	s.	Years.	£	Per cent.				
$\frac{1}{4}$	7	15	12	0·825	61·0	477	70	4·4	0·40
$\frac{1}{2}$	10	7	13	1·042	66·7	373	28·3	5·4	0·39
$\frac{3}{4}$	12	4	14	1·17	70·2	317	19·4	5·3	0·38
1·0	14	8	15	1·32	71·8	291	14·9	5·2	0·39
1·5	17	10	16	1·54	75·7	239	9·0	5·7	0·37
2	20	0		1·76	77·0	223	6·25	6·1	0·38
3	26	8	17	2·24	78·6	203	4·1	6·2	0·39
4	29	10		2·50	79·9	188	3·6	5·3	0·40
5	32	7	18	2·65	80·9	176	3·0	5·2	0·40
7·5	37	4		3·05	82·4	159	2·2	4·6	0·41
10	44	16	19	3·57	83·6	147	1·2	6·4	0·42
12·5	50	0		3·97	84·0	142	0·75	8·2	0·44
15	57	8	20	4·42	84·2	140	0·38	—	0·46

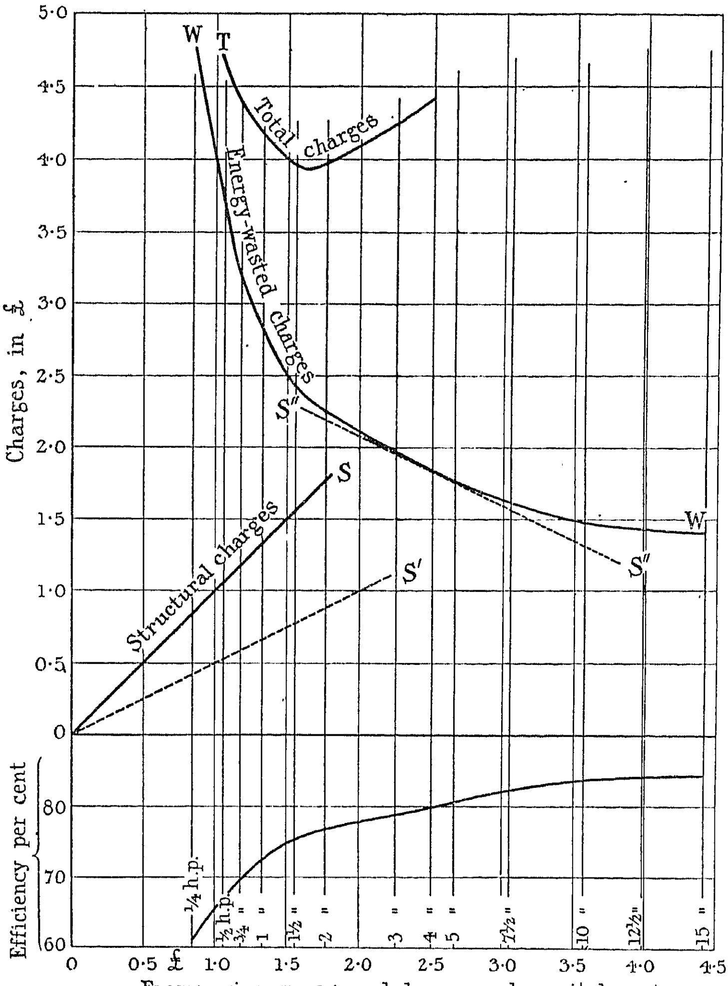


Fig. 7.—D.C. Motor Costs.

the power-service-price (P.S.P.),* the loss charge becomes $Q \times \frac{\text{P.S.P.}}{1{,}000}$ (£).

In Fig. 7, curve W is for a power-service-price of 10, *e.g.*, for a

* Thus for a 10-h.p. motor running for eight hours a day, 300 days a year, with energy at 1*d.* a unit, the P.S.P. would be $(10 \times 8 \times 300)/240 = 100$, and the same figures would cover 10 h.p. for twenty-four hours a day with energy ⅓*d.* and 40 h.p. for four hours a day with energy at ½*d.*

1-h.p. motor running eight hours a day for 300 days with energy at 1*d*. per unit, or 24 × 300 hours with energy at $\frac{1}{3}$*d*. per unit, or any equivalent product; and as the vertical scale is in £ the values for W are obtained from the loss ratio Q (column 6) divided by 100.

It is now possible to add together curves S and W, giving a total T which shows a minimum just above the frame rated for a maximum of $1\frac{1}{2}$ h.p. Thus for the output and service mentioned above, or for an even smaller power, if energy were more expensive it would be economically sound to purchase the $1\frac{1}{2}$-h.p. frame and under-run it to this extent. In the case of energy costing $\frac{1}{2}$*d*. or less per unit, and where the motor runs eight hours a day, it will be seen that this frame size would be economical for outputs of 2 h.p. or over—in other words the usual physical criteria would be justified as the basis of choice.

In the case of some other value of the power-service-price, say, 20 (*i.e.*, 2 h.p. for 8 × 300 hours at 1*d*. per unit, or its equivalent), the curve W should be re-plotted, in this case twice as high. To avoid doing this, the ordinate scale can be halved, which will have the effect of bringing the capital charge curve to a new inclination (see dotted line S′). Instead of adding this to the W curve it can be reversed and then brought up to the latter in order to see at what point it is tangential (see dotted line S″). Thus it will be noticed that at or near the 4-h.p. frame the slopes of the two are the same, thus showing the economical position for this particular power-service-price. (It is interesting to compare Fig. 7 with the corresponding graph for a cable, Fig. 5, p. 107.)

D.C. Motors: Algebraic Solution.—It has been seen that omitting all those items which are unaffected by the choice of frame size, the total annual expenses involved in obtaining the required service can be expressed as the sum of the capital and loss charges $Cr + Q \times \dfrac{\text{P.S.P.}}{1{,}000}$.

Where C is the machine first cost
r is the total rate for interest plus depreciation (adjusted, if necessary, for salvage value).
Q is the loss ratio.
P.S.P. is the power-service-price.

When these two are plotted to a base of C, the former is a straight line rising with C, whilst the latter falls as C increases,

since the efficiency is greater, and therefore the loss (for a given horse-power) less, the greater the expenditure on frame cost (*cf.* Fig. 8). The sum of the two will indicate the point of minimum cost and therefore the most economical size of frame to employ.

The only objection to this method of calculation is that it has to be repeated not only for every different horse-power but for every change in the conditions of service. In order to avoid such a recalculation and to establish a general method of solution for any machine of this type and speed, the total charge expressed above can be differentiated with respect to C, giving a rate of change $r +$ (P.S.P./1,000) $\times\ dQ/dC$, which must be zero at the point where the total charge is a minimum. Hence at this point $dQ/dC = -$ 1,000 r/P.S.P., or if the interest and depreciation are expressed per cent. (r'), $dQ/dC =$ $-$ 10r'/P.S.P. By finding the value of dQ/dC for a complete range of machines of the type concerned, it is possible to use this to determine the most economical frame size for any power and conditions of service whatsoever.

In the table of D.C. motors used in the last example, column 6 gives the value of the loss ratio Q, calculated from $(1/\eta - 1) \times 746$. In order to find the rate of change of this quantity with respect to C, it can either be plotted to this base on a large scale and the slope determined graphically, or else an empirical formula can be developed and then differentiated. In this case the graphical method was employed, and the value dQ/dC was thus found for each frame size and is shown in column 7. (The actual values are, of course, negative.)

This value is the key to the economic solution, and from it the most advantageous size can be estimated for any set of prices and conditions of service. Thus, to take a single instance, suppose that the service required is 1 h.p. for eight hours a day and 300 days a year with energy at 1*d.* a unit. The P.S.P. is $1 \times 2{,}400 \times 1/240 = 10$. Taking a life of fifteen years with interest at 5 per cent. and salvage value 10 per cent. of first cost, the combined rate for interest plus depreciation is 7·7 per cent., or $r = 0{\cdot}077$. Hence the most economical size will be when the rate of change $dQ/dC =$ 1,000 r/P.S.P. $=$ 7·7, and this will occur on a frame rated somewhere between $1\frac{1}{2}$ and 2 h.p. (column 7).

Referring again to the table, it is interesting to see whether the advantage of under-running is the same for all sizes. In order to make the values in column 7 mutually comparable, a

set of figures has therefore been worked out for lives of twelve to twenty years (column 3), with interest at 5 per cent. and salvage value 10 per cent. of first cost. These figures (column 8) give the value of $(dC/dQ) \times (1{,}000\ r/\text{Rated h.p.})$, and hence show the service-price at which each frame can economically be used to give its normal rated horse-power. It will be seen that the figures range from about 4 to 7, there being no discernible tendency for the figures to vary with the size.

A.C. Motors.—The next type of machine to be considered is that of a squirrel-cage induction motor running at 1,000 r.p.m. (synchronous) on a 50-period supply. Columns 2 and 3 in the table below give the first cost and the full-load efficiencies for a range of frames rated at 1 to 100 h.p. at this speed, the actual figures being averaged from quotations of five or six different makers. Column 4 gives the loss ratio, calculated from $(1/\eta - 1) \times 746$.

In finding the rate of change of this quantity with respect to C, two different methods were tried, namely, graphical differentiation by measurement of slope at various points and algebraic differentiation of an empirical formula representing the price.* The two methods gave results fairly similar over most of the range, but appreciably different at the lower extreme. In any actual problem, of course, the choice would only range over a comparatively few sizes, and it is probable that either the graphical or the algebraic differentiation would give equally satisfactory results. Using the graphical figures, column 5 gives the values of dQ/dC for each value of C which corresponds to an actual frame size.

In order to illustrate the use of the above figures, three cases may be given, differing as much as possible in their data. Tabulating first the data, and then the calculations therefrom, these are shown in the table on p. 149. The first step is to work out the power-service-prices (P.S.P.) and the combined rate of interest and depreciation per cent. (r'). The quotient $10r'/\text{P.S.P.}$ is then worked out, and from the previous table the size of frame is then found at which the rate of change, or slope, dQ/dC has this value.

It will be noted that in not one of the above cases can the correct frame size to employ be said to be definitely specified

* By taking logarithms of Q and C and plotting them, a very close approximation to a straight line was obtained over the above range. The empirical law thus established was $Q = (C)^{-0{\cdot}48}$, whence $dQ/dC = -0{\cdot}48\,(C)^{-1{\cdot}48}$.

L 2

Squirrel-Cage Induction Motors

Column 1.	Column 2.		Column 3.	Column 4.	Column 5.	Column 6.	Column 7.	Column 8.
						At 5 per cent. Interest.		
Rated Output.	First Cost, C.		Rated Full-load Efficiency, η.	Loss ratio, Q.	$-\frac{dQ}{dC}$.	And with life of	Economical Service-price.	$(1/\eta - 1) \times \sqrt[3]{\text{h.p.}}$
H.p.	£	s.	Per cent.			Years.		
1	12	18	77·0	223	12	15	7·6	0·30
2	14	10	78·7	202	9·5	16	4·6	0·34
3	16	4	80·0	186	6·3	17	4·5	0·36
5	22	4	82·0	164	2·95	18	5·6	0·37
7½	26	16	83·3	150	2·34	18	4·7	0·39
10	31	10	84·0	144	1·87	19	4·2	0·41
15	39	5	85·3	129	1·43	19	3·7	0·42
20	46	4	86·0	121	1·18	20	3·3	0·44
25	52	16	87·0	111	1·00	20	3·1	0·43
30	58	16	87·5	107	0·92	20	2·8	0·44
40	69	18	88·5	97	0·76	20	2·5	0·44
50	81	5	89·0	92	0·66	20	2·3	0·45
60	92	0	90·0	83	0·57	20	2·3	0·43
80	110	10	91·0	74	0·36	20	2·7	0·43
100	132	12	91·5	69	0·30	20	2·6	0·43

by the results of the economic calculation. In the first case, with continuous running and relatively dear energy, the economical size is enormously bigger than that determined by heating considerations, namely 15 h.p. for an output of 2 h.p. Naturally, so big a frame as the 15-h.p. size would lose its efficiency if under-run to anything like this extent, but the economic calculation is sufficient to show that for this service a very much larger size than the 2-h.p. frame would be justified.

	Case 1.	Case 2.	Case 3.
Horse-power required at 1,000 r.p.m.	2	40	90
Average hours per day during which the above horse-power is required.	24	8	4
Days per year	365	300	300
Price of energy per kWh	1*d.*	$\frac{1}{2}$*d.*	$\frac{1}{3}$*d.*
Power-Service-Price (P.S.P.)	73	200	150
Estimated life (years)	16	20	20
Estimated salvage value	Zero	10 per cent. of 1st cost	10 per cent. of 1st cost
Rate of interest on capital	$6\frac{1}{2}$ per cent.	5 per cent.	8 per cent.
Combined rate per cent. (r') of interest and depreciation.	10·24	7·7	9·97
Value of $10r'$/P.S.P. (Which must equal the value of the slope dQ/dC at the most economical point)	1·40	0·385	0·665
Rated h.p. of frame nearest to this point	15	80	50

For the 40-h.p. service also an appreciably larger size is indicated, probably 60, 70 or 80 h.p. rating, depending upon the alternatives actually available and the proportions of the losses (see below). In the case of the 90-h.p. service, owing to the small number of hours a day and the cheapness of the energy, combined with the high rate of interest, the economical size is smaller than that based on the ordinary heating rating, and the latter must therefore be chosen.

The above cases are illustrated graphically in Fig. 8. The base of this graph is the first cost C plotted to an even scale (£), and on this base are also shown the rated horse-powers of the frames obtainable at these prices. The reciprocals of the

efficiencies obtainable with these frames are represented by the falling curve labelled "loss ratio," the ordinates showing to some scale the quantity $(1/\eta - 1) \times 746$. For any particular service the loss charge (£) would therefore be obtained from

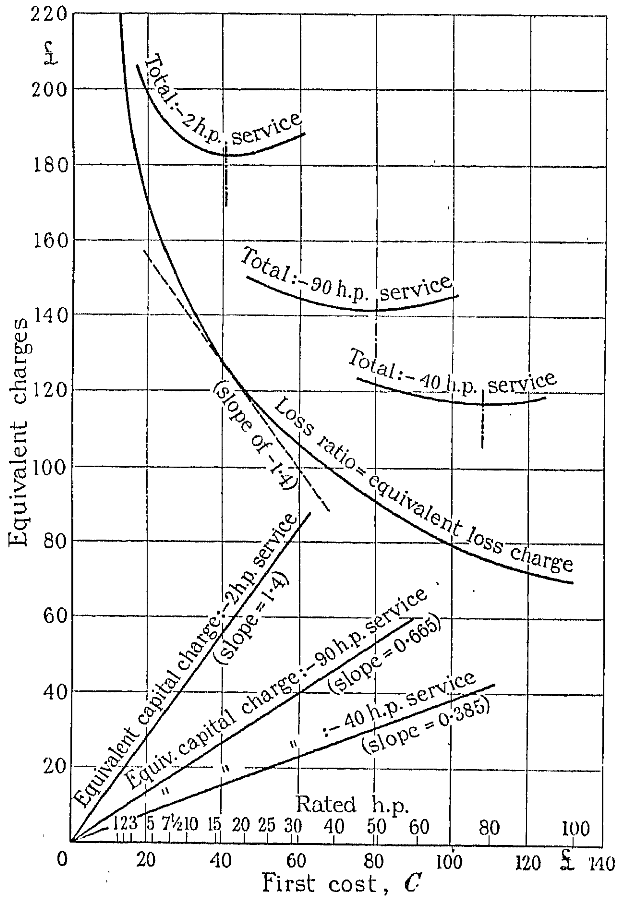


FIG. 8.—Squirrel-cage Motor Costs.

this curve by multiplying by P.S.P./1,000. On the same base the structural or capital charge Cr will be a straight line through the origin, and in any given instance the two curves can be added to see where the total becomes a minimum.

To avoid redrawing for each particular case, the ordinates can all be divided by P.S.P./1,000, the values then being

termed "equivalent charges" (see Fig. 8). The loss curve is then fixed for every condition of service, and the capital curve plots Cr divided by P.S.P./1 000 $= \dfrac{1{,}000\,Cr}{\text{P.S.P.}}$, which will still be a straight line through the origin, its inclination varying with every change in the conditions of service and the prices of energy and capital. In the three cases listed above, its inclination will be given by arc tan 1·4, 0·385 and 0·665 respectively, and these lines are labelled "equivalent capital charges." Adding these to the loss curve gives the three totals shown, and thus indicates the frame position for minimum total cost in each case. Another way is to reverse the capital curve and bring it up to the loss curve so as to show at what point it is tangential (see dotted line, representing slope of $-$ 1·4).

In a similar manner any other power, hours of service, or price of energy or capital, can be resolved into a factor which will at once show the economical size of frame on the assumptions made above. If the frame thus specified is not more than, say, 50 per cent. larger than the size determined by the heating limit, and if there is no possibility of qualitative selection, this size may be chosen without further calculation, care being taken to purchase a higher-voltage machine so that it may be under-run as regards pressure as well as current. If the economical size indicated is very considerably above the heating rating, further consideration must be given in view of the loss in efficiency occasioned by substantial under-running.

Thus in Case 2 given above (40-h.p. service) probably the best standard frame that could be chosen would be the 60-h.p. size, which would then be under-run in the ratio 1·5 to 1. Assuming that the full-load current, pressure and friction losses are in the ratio of 4 : 2 : 1, the most economical pressure reduction will be in the ratio $\sqrt{\dfrac{1\cdot5}{\sqrt{2}}} = 1\cdot03$ and the current reduction will be $\sqrt{2}$ times this, or 1·46, making a total power reduction of $1\cdot03 \times 1\cdot46 = 1\cdot5$ (Appendix IV.). Hence, if the supply pressure is 400 volts, the 60-h.p. machine can be ordered for, say, 412 volts, and when under-run to give 40 h.p. at the lower pressure it will be found to have an efficiency of 89·8 per cent. The annual cost for capital and losses will then be £24 as against a figure of £24 16*s*. for the 40-h.p. frame run at its

full output ; and this is apart from any incidental gain such as longer life or greater overload capacity. It will be noted that this case is similar to example (3) of the previous chapter, except that the service-price is different and a somewhat more favourable assumption has been made as to the proportions of the losses.

As in the previous case, a set of figures has been worked out assuming interest at 5 per cent., salvage value 10 per cent. of first cost, and lives varying, in this case, from fifteen to twenty years (columns 6 and 7). These figures show that the service-price at which the machines can be economically employed on their rated outputs declines roughly from five on the smaller sizes to $2\frac{1}{2}$ on the larger, and such values can only be described as extraordinarily small.* They are approximately half the corresponding figures for D.C. motors, implying twice as good a case for economic selection.

The above leads to the somewhat startling conclusion that for large squirrel-cage motors in use during an average working day the physical criteria by which alone they are at present selected become unsatisfactory whenever energy costs more than $\frac{1}{4}d.$ per unit. All cases at about these figures require to be considered on economical as well as physical grounds, and where energy costs $\frac{1}{2}d.$ or more per unit, or the motor is required for day and night service, a very much larger machine will usually prove the cheaper installation.

Other Factors : Conclusions.—It will be noted that no mention has been made of any of the incidental advantages which could be expected from the employment of larger or more efficient machines, such as lower working temperatures, and hence probably longer life and greater freedom from breakdowns. On the other hand, in estimating the life at a figure of fifteen or twenty years, there may sometimes be an error in the other direction, since although there is nothing to prevent the

* It may be objected that the above rate of interest is abnormally low, but a change in this rate by no means [illegible] change in the result, since the interest payment and [illegible] are oppositely affected (*i.e.*, the full first cost is not really owing for the whole of the working life, but is gradually paid off, or balanced by interest-bearing deposits into a sinking fund). In the present case, if interest were at 8 instead of 5 per cent., the figure of 5 for the economical service-price on the small frame sizes (17-year life) would become 6·9, whilst the figure of 2·5 for the larger sizes (20 year-life) would become 3·2. Hence it will be seen that, even if capital is considerably more expensive, the service-price at which these frames are economical for their rated outputs is still surprisingly small.

motor from lasting twice this, the circumstances of its use may change, and if for any reason it has to be replaced by one of a different type or size, its salvage value is likely to be small.

A more important point concerns motors which are not to be used at or near their full rated output during the time for which they are connected. In such a case the question arises as to what exactly constitutes the power of any particular service. In using the phrase "a 10-h.p. 1,000-r.p.m. service" it is assumed that 10 h.p. is the actual load whenever the motor is in use, or failing this the average load. Even if it is the former, many engineers would put in a 12-h.p. or 14-h.p. frame, and still more so if the 10-h.p. is an *average* output. Thus to speak of the 10-h.p. frame as the normal physical choice may be unfair to present-day choosers, many of whom might appear to be yielding to an unconscious economic sense in specifying a larger machine than is necessary (although if so the application should be more scientific and should endeavour to under-run both the iron and the copper).*

More usually, any under-running or over-specifying that may take place is with a view to possible overloads rather than on economic grounds, and it must be realised that the former is just as much a physical criterion as is the temperature rise. Under-running on economic grounds is entirely distinct and takes different forms; thus a 2-h.p. frame under-run to give 1 h.p. on the lines suggested would *not* (unless re-designed) be suitable for giving 2 h.p. even occasionally, since its copper loading would then be excessive. In the case of widely fluctuating loads it will be understood that although an approximate estimate might be made on the basis of the average load, this could not be considered very satisfactory, since the average efficiency of a machine whose copper loading alone was varied would be considerably below the maximum efficiency.

Hence on a varying load (since only the current loading will be varying) the required loss ratio must be calculated from the full-load pressure and frictional losses plus the R.M.S. value of the current losses divided by the mean output, and this can only be accurately determined if a load curve for the year is available. The point is not quite so important with motors as it is with lighting transformers, which are

* A reduction of the copper loading only i[illegible] be deprecated in the case of induction motors as this reduces [illegible]

frequently only on full load for a fraction of the time for which they are connected—a problem which has been considered by a number of writers on the subject of transformers. For such a case the methods of this and the next chapter cannot do more than indicate very approximately the total economic size, whilst throwing no light on the most economic proportions.

The above remarks apply with even more force to a further item which has been neglected, namely, varying costs of energy. When the cost of energy varies, the loading remaining constant, it is legitimate to take an average of the energy price over the period during which the plant is connected; but when the loading and therefore the efficiency vary simultaneously with the energy price this calculation will be inaccurate, since then the different losses will have varying degrees of importance according to the time of day at which they occur. Such a problem is considerably more complicated, and it is impossible to do more than mention it here.

Another item whose variations will affect the results somewhat is the rate of interest on capital. Electric motors having some degree of market value independently of the prosperity of the concerns in which they are used may rank amongst the tangible assets paid for by debenture shares, and in this case the rate of interest on extra capital so employed need not be high; on the other hand, if their purchase is considered to involve a risk and has to be financed by more speculative investments, the rate of interest will naturally be higher. Two points should be noticed in this connection: first, the capital charge is not *proportional* to the rate of interest, since the depreciation item is less when the rate is greater (see footnote, p. 152); and secondly, a general alteration in the market rate of interest is unlikely to make any ultimate difference to the above results, since the price of energy is also to a large extent a function of interest on capital.

Summarising the conclusions of this chapter, it may be said that most electrical machinery can be employed at an output lower than its rated one in such a way as to give an efficiency greater than that obtainable on a smaller machine running on its full load. This fact may be made use of on economical grounds, not merely as a possible basis for action in itself, but also as a guide to the still greater advantages possible through re-designing. The method developed enables the designer or purchaser to estimate the extent to which efficiency is worth paying for, or, more strictly, the quantities

of active materials which are economically justified, under any given set of conditions.

The results thus obtained show that for motors working chiefly at full load the employment of the smallest possible machine with the minimum of copper and iron to satisfy temperature requirements is justifiable economically only if energy is exceptionally cheap and/or service is for a small portion of the day only. The results also show that the advantage of under-running is no less marked on the larger than on the smaller powers, and would emphatically appear to call for a general review of this question and a detailed consideration of all sizes of electric motors, both D.C. and A.C. It is not too much to suggest that such a review may lead to a complete revision of our methods of choosing motors for long-period service, future ratings being determined by economic considerations, whilst the heating properties, like the insulating properties, fix only a lower limit.

CHAPTER IX

TRANSFORMERS

Efficiency.—The static transformer is in most respects similar to the D.C. and A.C. motor as regards the possibility of economic choice. Again, the only problem which need be considered is that of paying for efficiency, since the other economic differences—life, reliability, etc.—hardly exist as between makers of repute. There is, however, one feature of the transformer which should be mentioned in this respect, namely that the normal efficiency being already very high, there is much less room for improvement than there is with motors. Putting the same thing more precisely, it may be said that the inefficiency charge is a much smaller proportion of the total cost with transformers than it is in the case of most other electrical plant. These smaller losses are sometimes balanced by longer hours of service, but usually any such longer hours are not at full load, and, in general, it will be found that the advantage of paying for higher efficiency is less with transformers than it is with rotating machinery.

In order to illustrate this point, comparison may be made with the induction motor for some particular set of conditions. At the beginning of Chapter VII. a 40-h.p. squirrel-cage machine was instanced, costing £70, and in which the value of the losses *each year* came to £38 15*s.*—more than half the first cost of the motor. Taking (as a more typical size) a 100 kVA transformer on the same circuit, costing £110 and having an efficiency of 98·3 per cent., the value of the losses each year for the same hours of service and price of energy would be only £17 6*s.*—about 16 per cent. of the first cost of the transformer. (This comparison is more fully carried out in the table on p. 162.)

As with motors, it is sometimes possible to obtain alternative quotations from the same makers, or from rivals of equal repute, embodying transformers differing only in price and efficiency. The choice can then be made on the lines already indicated, and an example is worked out below, from which it will be seen that the method followed is almost exactly that of Chapter VI. (pp. 123 and 124). Failing alternative quotations of

this sort, it is still possible, with transformers as with motors, to under-run a larger machine of the same type, thus utilising what has been called the quantitative method of efficiency improvement. In fact, this will frequently be the only possible way, for with transformers the efficiency question has received a large amount of attention, and as a result most modern transformers already employ the highest grade iron, etc., and do not present much margin for qualitative improvement. Thus in the example given below, the alternatives (a) and (b) may actually be two different sizes (with the prices and efficiencies shown), instead of the same frame size with qualitative differences. Later on, the general possibilities of this type of selection are investigated over a whole range of sizes, in the manner of the last chapter.

As regards the actual practice of under-running, the transformer is at an advantage as compared with the rotating machine. With the latter, frictional losses will usually prevent the efficiency of the larger machine from being maintained at quite its full load value, but with transformers there are only the two groups of losses, so that by suitable proportioning the under-running there is no reason why the efficiency should not be maintained absolutely intact. Hence, in selecting from a range of standard ratings, each size can be regarded not as something capable of a particular output, but rather as something capable of a particular efficiency, whilst having, of course, an upper limit of output beyond which it must not be taken.

Another type of economic problem concerning transformer efficiencies occurs in assigning the best proportions of the two losses, but this is a problem for the designer or manufacturer rather than for the individual purchaser, and thus belongs to a part of the subject not dealt with in this book. The problem arises particularly with lighting transformers which are in circuit for twenty-four hours a day, but are only on full load for a small portion of the time. In such a case the iron losses are clearly more important than the copper, since they occur in full all the time the transformer is connected. Against this must often be put the fact that the copper losses chiefly occur at peak period when the energy costs more to manufacture. The whole subject has been dealt with very fully by a number of writers and will not be further touched on here.*

* See particularly a summary in "Science Abstracts," 1917, Vol. 20, Abstract No. 714, of a paper by L. Vidmar.

Particular Case.—Example 1. A 100 kVA transformer is required for service eight hours a day and 300 days a year at full load and unity power factor. Useful life is estimated at twenty years with a salvage value 5 per cent. of first cost, and interest is at 6 per cent. per annum. Two transformers (a) and (b) are put forward as shown in the first two lines of the table below, these being similar in every way except for the differences given. The problem is to find which will be the more economical when energy costs $1\frac{1}{4}d.$ a unit, and to find also the economical sphere of each transformer in terms of service-price.

The basis of comparison as before is to be annual costs, and, since the problem concerns a choice between two specific alternatives, it will be sufficient to tabulate all the relevant costs and compare the totals. The structural or capital charge is found by multiplying the first cost by the rate for interest and depreciation, namely, $0{\cdot}06 + \frac{95}{100} \times 0{\cdot}0272 = 0{\cdot}0858$, and this is shown in the next line of the table. The following line shows the ratio :—loss in watts/output in kW, which is called the loss ratio (Q), and is found from $(1/\eta - 1) \times 1{,}000$. The watts lost are therefore found in each case by multiplying Q by the output in kW. The service-price is $\frac{8 \times 300 \times 1\frac{1}{4}}{240} = 12{\cdot}5$, and as this measures the annual cost (£) per kW of loss, it follows that in this case the loss or inefficiency charge will be $Q \times 100 \times \frac{S.P}{1{,}000} = 1{\cdot}25\, Q$ £ per annum. This is the only relevant working cost in the problem, and is shown in the next line of the table.

Adding these two charges gives totals which are almost the same for the two transformers—to be precise, of £30 18*s.* and £31 1*s.*—so that the one with the larger first cost proves also to be slightly dearer in the long run, and, in any case, the difference is not sufficient to justify any departure from normal practice.

Since only two machines are in question, some work can be saved and greater accuracy of calculation can be obtained by considering only the difference between the two. Thus the extra cost of transformer (b) consists only of the difference in first cost multiplied by 0·0858, *i.e.*, 1·46 (£ p.a.). The extra

working charge of transformer (*a*) consists of the difference in loss ratios multiplied by $1\frac{1}{4}$, *i.e.*, 1·29 (£ p.a.). Hence by comparing these two differences the problem can be solved to a greater degree of accuracy, and this is fully shown in the third column of the table below.

100 kVA TRANSFORMER

Transformer.	(*a*).		(*b*)		Difference.	
First cost (C)	£93		£110		£17	
Full load efficiency (η)	98·2 per cent.		98·3 per cent.			
		£		£		£
Capital charge 0·0858 C	—	7·98	—	9·44	—	1·46
Loss ratio (Q)	18·33	—	17·30	—	1·03	—
Loss charge, $1\frac{1}{4}$ Q	—	22·91	—	21·62	—	1·29
Total relevant charge		30·89		31·06		

With regard to the second part of the above question, it is easy to see that for any service-price below $\frac{1\cdot46}{1\cdot03} \times 10 = 14\cdot2$ machine (*a*) will be cheaper, and only for service-prices above this will the extra first cost of machine (*b*) be justified. If the hours of service are fixed at the above figure this means that energy must cost nearly $1\frac{1}{2}d.$ a unit for machine (*b*) to be profitably employed. The above calculation emphasises what has already been said about the smaller justification for economic choice with transformers than with motors or cables. At the same time, there are plenty of cases of dear energy or long hour service when such choice will be very amply justified.

Choice over Range.*—In what follows it is proposed to make a general survey of the possibilities of economic choice of transformers precisely as was done for motors in the previous chapter. Most of the preliminary explanations there made apply exactly to the present case, and it will be assumed that the full load efficiencies quoted for the different sizes belong to those sizes (and prices) whatever the particular output required. As

* The data employed is taken from the author's paper, *Journal I.E.E.*, 1926, Vol. 64, No. 351.

was stated above, this is a more correct assumption in this case than it was for motors, owing to the absence of friction losses, so that the economic advantages of under-running can be pronounced upon with more certainty than with any other structure except the single loss type, such as the cable.

In the table below are shown the rated outputs, first costs (C) and full-load efficiencies (η) for a range of single-phase transformers on fifty periods, and for high-tension pressures not exceeding 2,000 to 3,000 volts. In this case the figures given are those of a single manufacturer and not averaged from a number, as it was desired also to compare the iron sections and other particulars. It will be noted that, high though the efficiency is, it goes up steadily with the size and price, but, of course, at a decreasing rate as the larger sizes are reached. Column 4 shows the loss ratio Q (= losses in watts/output in kVA), this being calculated from $(1/\eta - 1) \times 1{,}000$.

The procedure followed is exactly that of the previous chapter, but in this case the more general algebraic method will be employed first. It was seen on p. 146 that the point of maximum economy occurs when the rates of change of the two relevant charges are equal and opposite, from which it follows that at this point the rate of change $\dfrac{dQ}{dC} = -\dfrac{1{,}000r}{P.S.P.}$ where r is the combined rate of interest and depreciation, and P.S.P. is the product of the power output (kVA) and the service-price. Hence for any particular output, hours of service, price of energy and rate of interest, it is only necessary to work out the above quotient and then find out at what point the rate of change has this value.

By plotting Q to a base of C on a large scale and differentiating graphically the slope or rate of change $\dfrac{dQ}{dC}$ has been found for this range of transformers at all the points corresponding to actual frame sizes, and these values are shown in column 5. These figures furnish all the data which is required for any variety of full-load service within the range listed, and in which it is only proposed to select one of the actual sizes shown. If, however, it were desired to investigate possibilities lying between these sizes, it would be necessary to find the slope at intermediate points.

SINGLE-PHASE TRANSFORMERS

Column 1. Rated Output.	Column 2. First cost, C.		Column 3 Rated Full-load Efficiency, η.	Column 4. Loss ratio, Q.	Column 5. $-\frac{dQ}{dC}$.	Column 6. Economical Service-price at 5 per cent.; Life 20 years.	Column 7. $(1/\eta - 1) \times (\text{kVA})^{0.31}$
kVA	£	s.	Per cent.				
$\frac{1}{4}$	6	9	90·8	101·3	17	18·2	0·065
$\frac{1}{2}$	7	18	92·3	83·4	13	11·9	0·067
1	8	10	93·7	67·2	10	7·7	0·067
$1\frac{1}{2}$	10	12	94·49	58·3	5·2	9·9	0·066
2	12	4	95·18	50·65	3·3	11·7	0·063
3	15	6	96·17	39·83	1·95	13·2	0·056
5	20	0	96·40	37·35	1·02	15·1	0·062
$7\frac{1}{2}$	23	10	96·93	31·68	0·66	15·6	0·059
10	27	0	97·14	29·44	0·60	12·9	0·060
15	34	0	97·26	28·50	0·49	10·5	0·066
20	39	0	97·59	24·70	0·38	10·1	0·063
25	43	6	97·76	22·90	0·33	9·3	0·062
30	48	0	97·92	21·24	0·22	11·7	0·061
40	58	6	98·06	19·78	0·13	14·8	0·063
50	67	0	98·11	19·26	0·05	30·8	0·065
60	73	12	98·15	18·87	0·045	28·5	0·068
75	89	0	98·20	18·33	0·040	25·7	0·071
100	110	0	98·30	17·29	0·036	21·4	0·073

Example 2.—In order to illustrate the use of these figures it will be sufficient to take two cases, for the same output but under different service conditions. The output is to be 5 kVA at fifty periods, with a length of service of twenty years and a salvage value at the end of this time of 10 per cent. of the first cost. In the first case, let it be supposed that high prices rule, that energy costs 3*d.* a unit and capital 8 per cent. per annum, giving a combined rate for interest and depreciation of 9·97 per cent. If the service is required for eight hours a day and 300 days a year, the P.S.P. will be $5 \times 8 \times 300 \times \frac{3}{240} = 150$. Hence the most economical frame size will be when $\frac{dQ}{dC}$ has the value $-\frac{1{,}000r}{\text{P.S.P.}} = -\frac{99{\cdot}7}{150} = -0{\cdot}66$, which occurs at the frame rated to give 7·5 kVA.

Example 3.—As a contrast let it be supposed that the full-load service of 5 kVA is required day and night continuously * with energy at 0·7*d.* and interest at 5 per cent., giving a combined rate with depreciation of 7·7 per cent. For maximum economy $\frac{dQ}{dC}$ must now equal $-\frac{77 \times 240}{5 \times 24 \times 365 \times 0{\cdot}7} = -0{\cdot}6$, which occurs at the frame capable of 10 kVA (column 5).

Referring to this last example, if the intending purchaser, after working through to the above conclusion, wishes to follow the dictates of his economic conscience, his course will be to order a transformer of 10 kVA output and for a voltage $\sqrt{2}$ times his actual line pressure, so that if he requires 400 volts he will order approximately 565 volts. (The exact figure is not important; the one given assumes that the transformer is designed to give equality of losses at full load.) When employed to give 5 kVA at 400 volts, the copper and iron will then each be under-run in the ratio $1/\sqrt{2}$ and the two losses will be approximately halved, leaving the efficiency exactly as for 10 kVA, and the total annual costs a minimum.

Summary and Graphical Illustration.—As in the case of D.C. and A.C. motors, it is interesting to compare the results of under-running at different points in the range, and see whether there is any better justification for it on small transformers than

* It is realised that this is not a likely condition (see later).

on large ones, or *vice versâ*. Using some definite figures for the life, etc., it is possible to find what is the service-price at which each particular size is economical for its actual rated output. It will be seen from the equation on p. 146 that this service-price is given by $\frac{dC}{dQ} \times \frac{1{,}000r}{P}$, where P is the rated kVA output. Taking the same rates as for the motors, namely 5 per cent. interest and 10 per cent. salvage value, with a uniform life of twenty years (giving $r = 0{\cdot}077$), the values of $\frac{dC}{dQ} \times 77$/rated kVA are shown in column 6; and this column therefore shows the service-price at which each frame can be economically employed to give its actual rated output.

The figures are a little irregular, but show no tendency to vary, either way, with the size. Comparing these figures with the corresponding ones for the rotating machines, it will be noticed that the average for the transformers, namely fifteen, is three times as great as that for the small D.C. motors, and six times that for the A.C. machines. Hence the advantage of under-running is notably less, and the service would have to be at full load continuously in order to present as favourable a case for economic selection as with a D.C. machine running only eight hours a day. Another way of putting the matter is to say that, in order to establish an economic case for the installation of the next larger machine (say $1\frac{1}{3}$ times as big), the service price would have to be twenty—corresponding to energy at 2*d*. for an eight-hour day or $\frac{2}{3}$*d*. for a twenty-four-hour day.

It may be thought that the rate of interest assumed for the purposes of the above comparison is somewhat low, but as was explained earlier, a change in this will not make a *proportional* difference to the result. Thus if the interest were at 8 per cent. in the above example, all other items being unaltered, the average figure of fifteen for the economical service-price would then become twenty.

A graphical illustration of the foregoing work will be helpful, and this is shown in Fig. 9. The base of this shows first cost (£), but as regards the ordinates, no attempt has been made to scale these similarly. In fact, this graph omits the preliminary work of Fig. 8 in the previous chapter, and the ordinates are pure numbers, representing what may be called "equivalent

charges." The falling curve plots the loss ratio Q, and the straight lines through the origin plot the values $1{,}000r\,C/$P.S.P. The steeper of the two lines represents the 5-kVA eight-hour service at 3*d.* a unit, and has a slope of arc tan 0·66. The reverse of this line, having a slope of — 0·66, is also shown (dotted) brought up to the loss curve, and it will be seen to be tangential at the frame size rated to give 7·5 kVA. The shallower curve through the origin has a slope of just half as much and represents 10 kVA for the above-mentioned hours

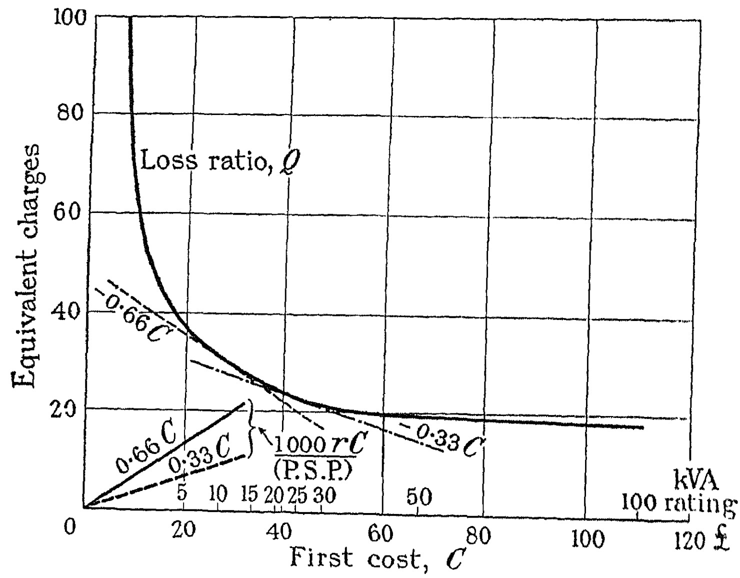


Fig. 9.—Single-phase Transformer Costs.

and price of energy. The reverse of this (chain-dotted) is tangential at the 25 kVA size and illustrates the economic point for this service.

In conclusion, it should be noted that in each of the examples taken the service has been at full load all the time the transformer is connected. When this assumption is not correct, even approximately, the problem becomes more difficult, and the remarks on p. 153 regarding motors in this position apply equally here. It will also be noted that unity power factor has been assumed throughout, and, whilst this may not be the case

in practice, the general relationship between the efficiencies of the different sizes is very similar whatever the power factor, provided the latter is not affected by the change in question. It is therefore doubtful whether an all-round alteration in power factor will materially affect the solutions obtained, but, of course, if the power factor is known it is an easy matter to use the appropriate efficiencies at this power factor throughout the calculation.

CHAPTER X

LAMPS

Economic Characteristics.—In the problems considered in this chapter, the service required is that of illumination of a given value and distribution, obtained from incandescent electric lamps. From the economic point of view, illumination differs in a number of respects from the other services considered in this book. With most of these there are two or three limits which may fix the size or type of apparatus to be employed—heating, regulation and economics. In a great many cases the first or possibly the second is reached before the third, and economics are not then a determining factor. But with lamps the economics are usually the only thing that matters, since the heating limit (as evidenced by an immediate burn out) is not likely to be reached, and regulation (as represented by the colour of the light) is rarely a determining feature. The choice may lie between lamps of different prices, the cheaper having a lower efficiency (*e.g.*, carbon or low-grade metal filament, as compared with a higher grade), or it may lie between lamps of different lives with corresponding differences either in efficiency or in price. In any case the choice has usually to be made purely on economic grounds, and technical considerations do not affect it, although, of course, they affect the data upon which the choice has to be made.

Another great difference between lamps and most other electrical apparatus is that the life is reckoned in hours of burning rather than in hours of actual existence, *i.e.*, the lamp is presumed not to depreciate except with use; and (barring mechanical vibration or long periods of idleness) this assumption is probably sufficiently accurate for the purpose. A further difference is that lamp renewals are relatively so frequent that the interest charges can be neglected in comparison with the depreciation charges.* This obviates the difficulty of having to summate replacement charges dependent

* A single example will show this. A lamp which is burning for three hours ght and lasts 1,000 hours has to be renewed approximately once a year. interest at 5 per cent. the interest charge would be 1/20th of the deprecia- r renewal charge, and can therefore be neglected in comparison.

upon hours of burning, and interest charges dependent upon calendar hours. As a result of these differences the economic calculations in connection with illumination are made on a basis of a service unit, *e.g.*, lumen-hours, whereas most other economic comparisons are made on the basis of a time unit, *e.g.*, annual costs or total capitalised costs.

The cost of a given quantity of illumination is made up of two main items—the cost of lamp renewals and the cost of energy. When these vary together, as in the choice mentioned below between different sizes of illumination unit, the question is not one that can be settled purely on economic grounds, although cost may be a very important item in the decision. But when the two costs are inversely related, the one going up as the other goes down, then a strictly economic choice becomes possible.

Choice of Size.—Before dealing with any of the purely economic choices mentioned above, it will be well to consider the question of the size of the illumination unit. This is usually fixed by the requirements of the particular case, *i.e.*, by the kind and distribution of lighting required, but in attempting to satisfy these requirements it is important to realise how very much cheaper a given quantity of illumination becomes when large units can be employed instead of small ones. Thus if a given area can be lit by either two 60-watt or three 40-watt lamps, the former arrangement will give more light for the money than the latter, and this for two reasons. The first is that the larger lamp is inherently more efficient than the smaller, so that if run to have the same length of life it will give more lumens per watt. The second reason is that only two lamps need be purchased per 1,000 hours instead of three, and each lamp costs the same. (In some cases the smaller power lamp actually costs more than the larger, owing to the cost of manufacturing the finer filament.)

In the following table is given a typical list of present-day prices and performances for vacuum and gas-filled lamps of British manufacture for 200- to 260-volt circuits. The efficiencies are the mean values throughout an average life of 1,000 hours, as specified by the British Engineering Standards Association.* It will be seen that the efficiencies go up steadily with the size, and at an increasing rate in the case of the gas-filled types, so that the largest lamp shown gives

* British Standard Specification No. 161, 1925. British Engineering Standards Association, 28, Victoria St., S.W. 1.

54 per cent. more lumens per watt than the smallest one. The smaller gas-filled lamps are little, if any, more efficient than the corresponding vacuum types, and have the disadvantage of a more concentrated filament, with, however, the advantage of slightly whiter light.

Size and Type.	Price. s. d.	Lumens per Watt.
20-watt vacuum	2 9	7·39
30- ,, ,,	2 6	7·76
40- ,, ,,	2 6	8·06
60- ,, ,,	2 6	8·27
60- ,, gas-filled	3 0	8·4
75- ,, ,,	4 0	9·0
100- ,, ,,	5 0	10·1
200- ,, ,,	9 0	11·4

With regard to prices, there is very little difference between the different sizes of vacuum lamps, and what difference there is, is in favour of the larger ones. The gas-filled lamps go up in price with size, but even here the increase is not in proportion to the increase in watts, and still less is it proportional to the increase in luminosity. Moreover, the prices of these larger sizes are probably not entirely represented by costs of manufacture, and may come down nearer to those of the lower sizes if the demand or other conditions change. (N.B.—Since this table was compiled, the 60 and 100-watt gas-filled lamps have been reduced slightly.)

Making use of this data, Fig. 10 shows the total cost of lamps plus energy involved in obtaining one million lumen-hours illumination, using the lamps shown and for the energy prices shown along the base. In order to cover a wide range, logarithmic scales have been employed for both ordinates and abscissæ, so that a given vertical distance represents a given ratio of multiplication. It will be seen that the curves of the 60-watt vacuum and the 200-watt gas-filled lamps are practically parallel, and their distance apart indicates that illumination carried out with the former costs about 37 per cent. more per lumen-hour than with the latter. The vacuum lamp curves converge somewhat, showing that the proportional saving due to large sizes decreases with an increase in the energy cost, although, of course, the absolute saving increases.

Comparing the 60-watt vacuum and gas-filled types, the latter is more expensive for energy prices under 6*d.*, but at this point the two curves cross, and with still dearer energy the gas-filled is very slightly cheaper. Comparing the biggest extremes shown, namely, the 20-watt vacuum and the 200-watt gas-filled, it will be found that the cost of illumination with the

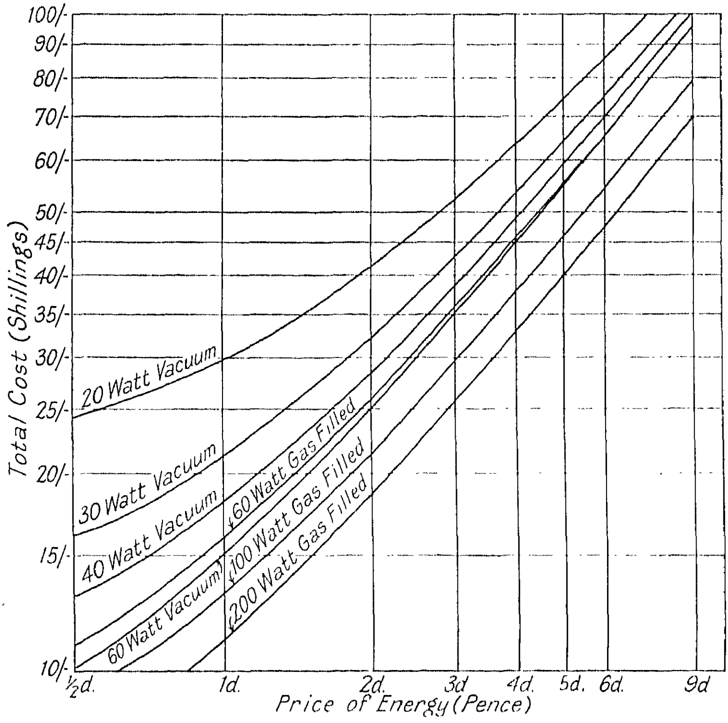


FIG. 10.—Cost of One Million Lumen-Hours (various lamps).

former varies from three times to one and a half times the cost with the latter, as the energy price goes up from $\frac{1}{2}$*d.* to 9*d.*

Choice of Type or Grade.—Using the word in its broadest sense, choice of type may be said to include any choice in which the type or grade of the alternatives is not known to be identical, *i.e.*, it covers not only the choice between carbon and metal filament, or vacuum and gas-filled, but also the choice between rival makes in which the efficiency or life are believed to be different. Omitting all qualitative physical differences

such as colour and filament shape, and assuming that the size of the illumination unit is approximately fixed, the difference will be seen to lie either in price, life or efficiency. Provided that two of these items are varying and in opposite senses (*i.e.*, the one favourably, the other unfavourably), a strictly economic choice becomes possible.

Example 1.*—As an illustration of the choice between lamps which are clearly of different types it will be sufficient to take a single case in which illumination is required in units having a mean horizontal candle-power of approximately 32 off a 220-volt circuit. It may be supposed that for this service there are available a 32-c.p. carbon lamp taking 110 watts and costing 1*s.* 9*d.*, a 40-watt vacuum metal-filament lamp costing 2*s.* 6*d.*, and a more efficient metal lamp, either vacuum or gas-filled, rated at 30 watts and costing 3*s.* 6*d.*† It is further assumed that they all give the same candle-power and that there is no qualitative difference between them for the purpose required.

Dividing the difference in price by the difference in watts, it will be seen that as between the carbon and the metal-filament vacuum lamp, energy will have to be less than $9d./70 = \frac{1}{8}d.$ per unit for the carbon lamp to be cheaper; whilst as between the two metal lamps, energy will have to be more than $12/10 = 1\frac{1}{5}d.$ per unit for the more efficient one to be cheaper. For any energy price between these two, the metal-filament vacuum lamp is the least expensive to employ.

The comparison would, of course, be more favourable to the carbon lamp if smaller lighting units were required, but however cheap the energy it is doubtful whether the carbon lamp would compete with a metal-filament lamp under-run in the manner described below, except under special circumstances such as severe vibration or extremely intermittent use.

The other kind of choice which may arise is between rival makes of lamp, *e.g.*, between British and foreign manufacture, when one of them is cheaper than the other, but is believed to be either less efficient or shorter lived or both. Unfortunately adequate data is rarely available, and as regards the life is

* This example and the sections which follow are taken from the author's paper, *Journal I.E.E.*, 1926, Vol. 64, No. 356.

† These figures are purely for illustration purposes to meet the case in which several makes or types of lamp are available for the same service, having different prices and corresponding efficiencies. On the larger sizes this often arises between the vacuum and the gas-filled types, but for the size mentioned gas-filled lamp has little or no advantage.

unobtainable except as the result of a large number of tests. When the lives are similar, but the efficiencies and prices differ and are both known, the problem becomes identical with that treated above. Examples in which the lives are different are given in the supplementary questions at the end of the chapter.

Choice of Rating.—In the cases mentioned above, the choice lay between two (or possibly three) specific alternatives whose characteristics were known. In the cases now to be considered, the choice lies between different ways of rating the same lamp or type of lamp. Hence the choice may be said to be over a whole range of alternatives instead of merely between two or three.

It was seen above that the cost of a given quantity of illumination is made up of two main items—the cost of lamp renewals and the cost of energy. Usually the latter represents from two to ten times the former, so that at first sight an improved efficiency would appear to be well worth paying for, even at the cost of a considerably higher lamp price. Unfortunately the only way of getting a better efficiency with a given type of lamp is by running it at a higher temperature with a correspondingly shorter life. As, moreover, a small degree of over-running, say, to 1 per cent. above the rated voltage, only produces about 2 per cent. increased efficiency, whilst it results in about 14 per cent. shorter life, it is evident that there are very narrow limits to the amount of over-running which is economically desirable.

Hence at the normal rating, when the energy and lamp renewal costs are in the ratio of, say, seven to one, a 1 per cent. increase in voltage will reduce the former by 2 per cent. (a decrease of 0·14 units of cost), and will increase the latter component by 14 per cent. (an increase of 0·14 cost units), leaving the total cost exactly as before. This, however, is only true for a given size and set of prices, so that whilst the existing rating may be correct for the average lamp at the average prices, it is by no means so for the exceptional cases. When energy is expensive compared with lamps it will clearly pay to over-run the lamp and *vice versâ*.

It will be seen that the rating of an incandescent lamp is settled not by physical or engineering considerations, but is purely a question of economics, since any desired efficiency can be obtained by suitably over- or under-running, and the efficiency normally quoted is merely that which the lamp will

attain when so run as to have a certain average length of life. This point of running, with the consequent length of life, is fixed by the lamp makers, usually at such a point that the lamps will have an average life of about 1,000 hours, but strictly it should vary according to the particular situation.

In considering problems concerned with the choice of rating, it is first necessary to fix on a criterion of rating. When a lamp which is intended by the makers to be used on a certain pressure is run on a different pressure, say a higher one, it may be said to be over-run, with results that can be summarised as (1) higher pressure, (2) higher efficiency, (3) higher candle-power, (4) lower life. The degree of over-running can therefore be expressed in the terms of the change in any one of these four quantities; and as the percentage efficiency change is very roughly twice the percentage pressure change, it will be convenient to define the degree of over- or under-running from the change either in efficiency or in applied pressure.

Graphical Treatment.—In what follows it will be best first to take a single instance, treating it graphically and approximately, before working out the more general case by an algebraic means, and for this purpose the following data will be employed :—

Size of illumination unit required—approx. 30 c.p.
Price of 100-volt lamp of this size—2*s.*
Normal rating—consumption 40 watts, efficiency 10 lumens per watt,* life 1,000 burning hours.

It is proposed to consider the effect of running this lamp at various efficiencies ranging from 10 per cent. below to 10 per cent. above the normal (column 1 of the table below), and this variation can be carried out in a number of ways. Thus a change in applied pressure of about half as much (say from 95 to 105 volts) will produce the required effect (column 2), and the same result would be achieved by a change in filament diameter from 90 per cent. to 110 per cent. of the normal, or in filament length from 105 per cent. to 95 per cent.†

Any one of these changes would affect also the candle-power of the lamp, but by changing the diameter and length simultaneously, the desired efficiency can be obtained without altering

* This figure is somewhat above present-day efficiencies for this size of lamp, but it will be convenient to have a round number for illustration purposes.

† These figures are approximate.

the size of the illumination unit. It may be assumed that, whatever the means taken to vary the efficiency, the effect on the life will be the same ; and most of the published figures give the life of a tungsten lamp as being inversely proportional to the efficiency raised to a power between 6 and 7. The value of this index is discussed later, but in this case it will be taken as having the value 6 (column 3).

40-WATT LAMP

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.
			Per 400,000 Lumen-hours' Illumination.		
Efficiency (η) (Normal = 10).	Applied Voltage necessary to obtain this with given Lamp. (Normal = 100).	Life (L) $\propto \left(\frac{1}{\eta}\right)^6$ (Normal = 1,000).	Cost of Lamp Renewals $\frac{1,000}{L} \times 2s.$	Cost of Energy at 3*d*. per unit. $\frac{100}{\eta}$	Total Cost.
Lumens/watt.	Volts.	Hours.	Shillings.	Shillings.	Shillings.
9	95	1,885	1·06	11·11	12·17
9·2	96	1,650	1·21	10·87	12·08
9·4	97	1,450	1·38	10·64	12·02
9·6	98	1 277	1·57	10·42	11·99
9·8	99	1,129	1·77	10·20	11·97
10	100	1,000	2·00	10·0	12·0
10·2	101	888	2·254	9·804	12·06
10·4	102	791	2·53	9·61	12·14
10·6	103	705	2·838	9·434	12·27
10·8	104	630	3·17	9·26	12·43
11·0	105	563	3·55	9·09	12·64

Taking as a basis the cost per 1,000 hours of one 30-c.p. unit (*i.e.*, per 400,000 lumen-hours), the cost of lamp renewals in shillings will be 2,000 ÷ (life in hours) (column 4), assuming that the 2*s*. paid per lamp includes any costs incurred in putting it in. With regard to the power consumed, when the filament has been adjusted to give the same candle-power, the watts will be 40 divided by the proportionate change in efficiency, and with energy at 3*d*. the cost for 1,000 hours will be $40 \times 10/\eta \times 3/12 = 100/\eta$ shillings (column 5). Adding the last two columns gives the total cost for this quantity of illumination, the minimum cost occurring at the point corresponding to an efficiency of 9·7 lumens per watt. With cheaper energy the minimum would occur earlier (*i.e.*, with the lamp more under-

run), and with dearer energy it would be later. This is shown in Fig. 11 to a base of efficiency, and having for its centre line

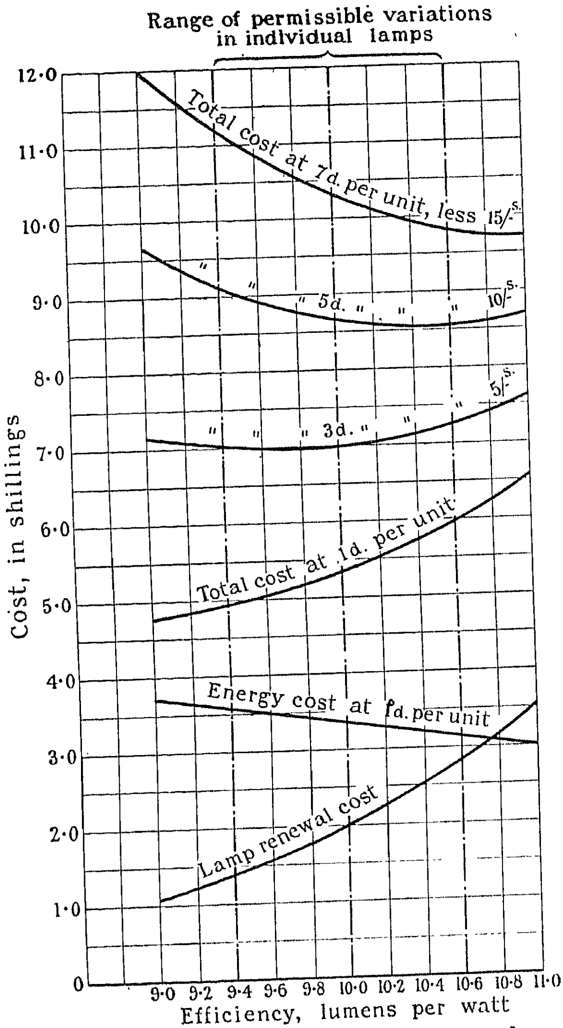


FIG. 11.—Economic Rating. (Cost of 400,000 lumen-hours with 40-watt lamps.)

the normal figure of 10 lumens per watt. The energy cost is a falling straight line whose slope depends on the price per unit, the lamp renewal cost is a rising curve, and

the figure shows the total cost for energy prices from 1*d.* to 7*d.* per unit. By drawing a tangent to the lamp renewal curve through its mid-point, the slope at this point is found to correspond to an energy price of 3½*d.*, showing that with energy at this price the lamp is most economical at its rated voltage.

Algebraic Solution.—In working out a general algebraic solution of the above problem it is desirable to know as accurately as possible how the various quantities, and particularly the life, vary with the change of rating. Unfortunately, there is considerable divergence in the published data, and it would appear that the simple formula—

$$\text{Life} \propto (\text{Voltage})^{\text{CONSTANT}}$$

is not true for wide variations, nor is the constant the same for different starting temperatures.

This point is chiefly important when excessive over-running is proposed, as when the average life at normal pressure is estimated from a series of "forced" life tests taken at considerably higher pressures. For economic purposes the degree of over- or under-running is much less than this, and any departure from the simple logarithmic form can probably be safely neglected in comparison with the very wide variations of the other items (cost of energy, etc.) in the problem.

The following symbols will be employed :—

V = impressed voltage
F = luminous flux (in lumens)
W = watts consumed
η = efficiency (in lumens per watt)
L = life (in 1,000 hours)
} with suffix n to denote the normal values.

C = first cost of lamp
P = price of energy per kWh
} in the same units.

It is assumed that all quantities vary according to some simple power of the impressed voltage, and if the latter is varied in the ratio R then

$$\frac{F}{F_n} = \left(\frac{V}{V_n}\right)^a = R^a\,;\quad \frac{W}{W_n} = R^b\,;\quad \frac{\eta}{\eta_n} = R^c \text{ and } \frac{L_n}{L} = R^d\,*$$

where a, b, c and d are fixed constants, provided R is restricted to a fairly small range (say from 0·9 to 1·1). Moreover, as $\eta = F/W$, it follows that c must equal $(a - b)$. In what

* This item is reversed because the life is inversely dependent upon the pressure, and it will be convenient to have all the indices positive.

follows the values will be taken as $a = 3{\cdot}6$, $b = 1{\cdot}5$, $c = 2{\cdot}1$ and $d = 14$ for vacuum tungsten lamps.

At the normal rating, for an illumination of F_n lumens lasting 1,000 hours the number of lamps required $= 1/L_n$ and the lamp cost $= C/L_n$. The watts consumed $= W_n$ or F_n/η_n and the energy cost $= PF_n/\eta_n$. Hence the total cost for 1,000 F_n lumen-hours, $T = (C/L_n) + (PF_n/\eta_n)$. Or the total cost for 1,000 lumen-hours, $T' = (C/L_nF_n) + (P/\eta_n)$.

Case 1. *When the same lamps are used, the pressure being varied.*—When the pressure is varied in the ratio R, each lamp will give a flux $F = F_nR^a$ and will have an efficiency $\eta = \eta_nR^c$ and a life $L = L_nR^{-d}$. The total cost per 1,000 lumen-hours then becomes

$$T' = \frac{C}{LF} + \frac{P}{\eta} = \frac{C}{L_nR^{-d}F_nR^a} + \frac{P}{\eta_nR^c} = \frac{C}{L_nF_n}R^{(d-a)} + \frac{P}{\eta_n}R^{-c}$$

In order to find what ratio of voltage variation will give the minimum total cost, it is only necessary to differentiate T' with respect to R, giving

$$\frac{dT'}{dR} = (d-a)\left[\frac{C}{L_nF_n}R^{(d-a-1)}\right] - c\left[\frac{P}{\eta_n}R^{-(c+1)}\right]$$

This is zero when

$$R^{(d-a+c)} \text{ or } R^{(d-b)} = \frac{c}{d-a}\cdot\frac{P}{C}\cdot\frac{L_nF_n}{\eta_n}$$

$$= \frac{c}{d-a}\cdot\frac{PL_nW_n}{C}$$

Putting in the above values for the constants, this becomes

$$R^{12\cdot5} = 0{\cdot}202\frac{PW_n}{C}$$

for lamps having a normal life of 1,000 hours.

It will be noted that the above case does not give a true economic comparison, because it results in the illumination being obtained in units of a new size. Thus if 30-watt lamps were proposed and the above calculation showed that the total illumination required would be most economically obtained by over-running the lamps, this might result in the flux per lamp equalling the normal output of a 40-watt lamp. But had it been known that the illumination could permissibly be carried out in the bigger units a saving could have been effected even

without over-running, both because of the reduced number of lamps required of the 40-watt size and because of the intrinsically better efficiency of the larger unit.

Case 2. With a fixed size of illumination unit (giving F_n lumens).—In this case it will be best to work on the basis of 1,000 F_n lumen-hours, the total cost for which is given above as $T = (C/L_n) + (PF_n/\eta_n)$ at the normal rating. When the rating is varied (to an extent equivalent to a pressure change of R) the filament dimensions are so altered that each lamp continues to give F_n lumens, and only the efficiency changes. The number of lamps required at any instant will be the same as before, so that the lamps required per 1,000 hours will still be the reciprocal of the life, and the total cost per 1,000 F_n lumen-hours

$$T = \frac{C}{L} + \frac{PF_n}{\eta} = \frac{CR^d}{L_n} + \frac{PF_n}{\eta_n R^c}$$

Differentiating, this gives

$$\frac{dT}{dR} = d\left[\frac{C}{L_n}R^{(d-1)}\right] - c\left[\frac{PF_n}{\eta_n}R^{-(c+1)}\right]$$

This is zero when

$$R^{(c+d)} = \frac{c}{d}\cdot\frac{P}{C}\cdot\frac{F_n L_n}{\eta_n} = \frac{c}{d}\cdot\frac{P}{C}W_n L_n$$

Putting in the above values for the constants, this becomes

$$R^{16\cdot 1} = 0\cdot 15\frac{P}{C}W_n L_n = 0\cdot 15\frac{PW_n}{C}$$

for lamps having a normal life of 1,000 hours.

This formula can be explained as follows with reference to the curves already plotted. Maximum economy occurs when the slopes of the curves of energy cost and lamp cost are equal and opposite. If these are plotted to a base of rating measured in efficiency, the former will be a straight line and the latter a curve the shape of which depends on the efficiency/life index (c/d). The slopes will therefore depend upon the values of these indices and also on the multiplying constants—so that *at any one rating*, maximum economy will occur when a particular ratio exists between the money normally spent on lamps ($\propto C/L_n$) and that spent on energy ($\propto PW_n$). This explains the appearance in the formula of items representing not only the energy and lamp prices, but also the size of lamp.

Application.—Taking the simplest form of the above, with constants for a vacuum tungsten lamp, namely, $R^{16 \cdot 1} = 0 \cdot 15PW_n/C$, this can be applied to the case already considered of lamps costing 2*s*. each and having a normal rating of 40 watts. For a fixed size of illumination unit (which will be about 30 mean spherical candle-power for a 100-volt lamp) the most economical point at which to run will be given :—

With energy at 1*d*., by $R^{16 \cdot 1} = \dfrac{0 \cdot 15 \times 1 \times 40}{2 \times 12}$, whence $R = 0 \cdot 917$, *i.e.*, 91·7 per cent. of normal pressure, or approx. 83·4 per cent. normal efficiency.

With energy at 4*d*., by $R^{16 \cdot 1} = 4/4$, whence $R =$ unity, *i.e.*, 100 per cent. of normal pressure, and efficiency.

With energy at 7*d*., by $R^{16 \cdot 1} = 7/4$, whence $R = 1 \cdot 035$, *i.e.*, 103·5 per cent. of normal pressure, or approx. 107 per cent. normal efficiency.

These agree fairly closely with the graphical points, allowing for the fact that in the graphs the index c is taken as 2 and the ratio d/c as 6.*

In considering the conclusions to be drawn from the above formula, it should be noted in the first place that of the three variables, the lamp prices vary least, *e.g.*, from about 2*s*. to 2*s*. 6*d*. for vacuum lamps, the lamp sizes vary more—say from 10 or 20 to 60 watts for these lamps—whilst the energy prices vary most, since they regularly range from 1*d*. to 8*d*. a unit or even more. As a rule the variations in lamp prices can be neglected or grouped with one of the other two variables, but as regards these latter, each must be considered separately, *i.e.*, in considering lamp size, the energy price must be presumed fixed and *vice versâ*. It will be noted that the variation of rating with size is a question for the lamp manufacturers or the Standards Association, whereas the variations to suit different energy costs is necessarily more a matter for the individual consumer.

* With these values for the indices, the normal rating coincides with the most economical one for an energy price of 3¼*d*., and for other energy prices the formula gives the most economical rating as follows :—

Energy at 1*d*.—Rating should be 91·2 per cent. of normal pressure or 82·5 per cent. of normal efficiency.

Energy at 3*d*.—Rating should be 98·7 per cent. of normal pressure or 97·4 per cent. of normal efficiency.

Energy at 5*d*.—Rating should be 102·6 per cent. of normal pressure or 105·2 per cent. of normal efficiency.

Energy at 7*d*.—Rating should be 104·7 per cent. of normal pressure or 109·4 per cent. of normal efficiency.

Variation with Size.—As regards the second of the three variables mentioned above, namely, lamp size, it will be seen that the larger the lamp, unless its price goes up in proportion, the more it should be over-run and the shorter should be the average life aimed at. Taking as a starting point the 40-watt lamp costing 2*s*., which is economical on its present-day rating with energy at 4*d*. a unit, it is evident that with this energy price, any lamp in which the lamp size in watts divided by the price in shillings exceeds 20 should be over-run. With a 60-watt lamp costing 2*s*. 3*d*., $R^{16 \cdot 1}$ should equal $1\frac{1}{3}$, whence $R = 1 \cdot 018$ and, as the life $L = L_n R^{-14}$, the best life will be $1{,}000/1 \cdot 018^{14} = 882$ hours. On the other hand, with a 20-watt lamp costing, say, 2*s*. 6*d*., the most economical life (with energy at 4*d*.) will be 2,230 hours. (In these calculations and in those of the table below, it is assumed that the filament dimensions are altered so as to give the same luminous flux as before.)

If the lamp price were proportional to the size in watts these items would, of course, cancel out, but, as mentioned above, the price varies little (and sometimes inversely) with the size. If lamp and energy prices are both constant, and if 1,000 hours is the most appropriate life for a 40-watt lamp, then the 20-watt size would have rather less than double this life and the 60-watt rather more than two-thirds.

Taking 4*d*. per unit as an average price for lighting energy to the small consumer, the following table gives approximate values for the most economical life for vacuum lamps of different sizes when purchased at the list prices shown.* The table also shows the corresponding efficiencies as percentages of the present values :—

Size.	120–130 Volt.			220–260 Volt.		
	Price.	Economic Life (hours).	Economic Efficiency per cent. of present value.	Price.	Economic Life (hours).	Economic Efficiency per cent. of present value.
	s. *d.*			*s.* *d.*		
20-watt	2 2	1,960	90	2 9	2,400	88
30- ,,	,,	1,380	95	2 6	1,560	94
40- ,,	,,	1,070	99	,,	1,210	97
60- ,,	,,	750	104	,,	850	102

* This table and the illustrations which follow are reprinted from *The Electrician*, March 11th, 1927.

It will be seen that at the above prices the 40-watt size of lamp is most economical at very nearly its present rating, but the others should be adjusted in the manner shown. Unfortunately, this will have the effect of increasing the discrepancy in efficiency between the different sizes. At present the larger lamps have an inherently better efficiency than the smaller ones (*i.e.*, at a uniform rating as regards life), and with the proposed re-rating these differences would be still further increased.

No doubt it was this tendency for the smaller sizes to lag behind in the matter of efficiency which led in the past to the adoption of a uniform lamp life, since to permit a further reduction in efficiency in the smaller lamps (even though coupled with the demand for a better life performance) might appear to be a retrograde step. Now, however, that the manufacture and performance of these lamps is so much better understood, it would appear that the time has arrived for a general reconsideration of these ratings, since there is no justification whatever for having a uniform life for all sizes, and there is no reason why consumers on average energy prices should not be given the benefit of the economy which would result.

Variation with Energy Price.—As regards the other main variable in the formula, energy price, there are much greater difficulties in the application of economic principles, both because of the wide differences to be found in this country and because of the fact that it is largely the lamp user (generally not a trained engineer) who has to take action in this case. There are, of course, ways in which the manufacturers could help, *e.g.*, by grading their lamps for dear, medium and cheap energy, somewhat on the lines of the "three-voltage rating" suggested some years ago in competing with the carbon lamp for cheap energies.* It will be noted that the B.E.S.A. Specification permits deviations in efficiency of about 6 per cent. on either side of the specified normal—represented by the upright lines on the graph, Fig. 11—and, when these variations occur, the consumer (on specially cheap or dear circuits) could be given the opportunity of choosing them in place of the normal lamps.

If no such help is forthcoming for the lamp user he must

* See especially, R. W. Hutchinson, "High-Efficiency Electric Illuminants."

needs act for himself, by selecting the most suitable lamp available for his particular local conditions. Referring again to the formula, it will be seen that for any one size of lamp the most economical rating will occur when a particular ratio exists between the energy and lamp prices. For a 40-watt lamp this ratio is one-sixth, *e.g.* :—

Energy at 5*d*. and lamps at 2*s*. 6*d*.
Energy at 4*d*. and lamps at 2*s*.
Energy at 3½*d*. and lamps at 1*s*. 9*d*., etc.

For a 60-watt lamp the ratio should be one-ninth, and for a 20-watt one-third.

When the ratio actually existing is fairly near to the above figure no action is necessary, but when the ratio is considerably above or below the appropriate one, due usually to specially dear or cheap energy, it will frequently be worth while to over- or under-run by the installation of lamps designed for a somewhat lower or higher voltage than the one on which they are to be used. One objection to doing this is that the candle-power is also affected, but this is not necessarily a disadvantage, since there is no special virtue in the particular candle-powers associated with 30, 40 and 60 watts, and in any given case some intermediate values may be as good or better. Nevertheless, in order to achieve a really appreciable economy without upsetting too much the existing sizes of the illumination units, such action is chiefly worth while when the departure from the appropriate price ratio is so great as to justify the substitution of the next size of lamp. When applied in cases in which the departure from normal prices is not so great, it will usually result in an appreciable increase in illumination, but not necessarily any decrease in costs.

Two examples may be given, the one involving under-running and the other over-running, in order to achieve the most economical results. In the first place, if the circuit voltage is 110 and lamps of about the 40-watt size are desired, the normal flux from such a lamp is 370 lumens. Assuming that 2*s*. is the price of the lamp, whatever the size or rating, the above calculation shows that with energy at 1*d*. a unit the most economical rating is that corresponding to 91·7 per cent. of the normal voltage, or about 73 per cent. of the normal candle-power. This can be exactly effected by fitting 120-volt lamps, but if, say, the 60-watt size is chosen so as to compensate for the lower voltage, the lamp flux will be 415 lumens instead of 370.

Taking as the other extreme a circuit with energy at *8d.* a unit, the calculation shows an economic rating of 104·4 per cent. of the normal voltage. This requires lamps designed for a pressure of about 105 volts, and employing the next smaller size (so as to compensate for the voltage change), a 30-watt lamp of this type will then give 320 lumens when run on 110 volts. Such small changes in the size of the illumination unit, although not necessarily disadvantageous, will slightly modify the economic calculation, since this is based on the assumption that lamps are available giving the original candle-power at the new rating, and having the same inherent efficiency as the ones they are to replace. It is therefore better, instead of working out the theoretically ideal rating from the formula, to use this latter merely as a rough guide, selecting by its aid some particular existing lamp and then working out the actual lumens and cost in the manner shown below.

Possible Saving.—Turning now to actual conditions, there are two main groups of cases in which there is likely to be a considerable departure from the appropriate ratio of energy price to lamp cost. The very big consumer, who will hardly be able to purchase lamps at less than, say, two-thirds of the listed price, can usually purchase his energy at a quarter or less of the average consumer's tariff. He should therefore usually under-run his lamps, and a typical case is worked out below. At the other extreme is the consumer, usually on a small scale, who whilst he pays the ordinary list price for his lamps, has to purchase his energy at a specially high figure, owing to his living in the country or being unfortunate in the matter of his supply authority. His most economical plan will then be to over-run the lamps so as to reduce his costly energy bill at the expense of his much smaller lamp account.

Objection has been made to the action suggested above on the grounds of the complication involved in the necessary calculations, and the small amount of the probable saving. Although these may be valid in the case of the second group mentioned above (small consumers on dear energy circuits), neither of these objections can be considered serious in the case of the former group. There are many big lamp users to whom a saving, even if it were small in proportion, would amount to a very considerable item. The underground railways employ tens of thousands of small vacuum lamps in lighting their stations, and these are burning in many cases for sixteen or

more hours a day. Big stores, hotels, hospitals, etc., and the municipalities in their street lighting are other examples of large-scale users.* Moreover, in almost all such cases, not only is energy purchased at very special rates (in some cases generated by the lamp users themselves), but also the maintenance is carried out by qualified engineers who can, without difficulty, work out the best lamps to use, and make the necessary changes in the stock and specifications.

Saving on Cheap Energy Circuits.—In order to see exactly how such a plan is likely to work out in practice and the saving which can reasonably be expected, it will be well to consider a typical large-scale user of small size lamps on, say, a 220-volt circuit. Let it be supposed that he can purchase energy for 1*d*. a unit, lamps at 30 per cent. off the list prices shown above, and that he is using 20, 30 and 40-watt vacuum lamps. Strictly speaking, the most economical degree of under-running will differ in the three cases, but for simplicity it may be supposed that he installs 240-volt lamps throughout. And in order to compensate for the reduced candle-power he changes his sizes to 30, 40 and 60 watt respectively. The 240-volt lamps, when run on 220 volts, will have a mean life of nearly 3,400 hours, and the following table gives the cost of 1,000 hours' illumination in each case. (This will be the annual cost when the average daily service is two and three-quarter hours.) The table also gives the lumens which he will obtain, these being calculated from the B.E.S.A. figures for the mean values over the lamp's total life.

In every case the cost is reduced, the reduction being as much as 24 per cent. on the smallest size of lamp, and in every case the illumination is more or less increased, the extra being 12 per cent. on the biggest size. If he is using 1,000 lamps of each size for an average of two and three-quarter hours a day, his yearly saving will amount to £28 (14·3 per cent. of his total illumination bill), and this in spite of an overall increase in illumination of more than 9 per cent. Since the calculation need only be made once, whilst the saving goes on from year to year, and since the amount of stock and difficulties of obtaining are not in any way increased, it will hardly be

* According to figures published by the L.C.C., the sum of £591,000 was spent in 1923 in lighting London's streets, and even if only a small proportion was electrical, and if the possible saving were only a fraction of that indicated in the examples below, it might still amount to some thousands of pounds a year.

suggested that this practice is not worth carrying out l large consumer.

If the ratio between energy and lamp prices is still low even more effective case for under-running can be pres Thus with energy at $\frac{1}{3}d.$ and lamps at the above disc 250-volt lamps can be employed on the 220-volt circuit, an 60-watt size will then give almost the same illumination 40-watt lamp on its normal rating, and at a total cost of

Lamps originally employed (220 v.)	20-watt	30-watt	40
Lumens originally obtained .	148	233	322
Lamp renewal cost (pence) . .	23	21	21
Energy cost (pence) . . .	20	30	40
Total cost (pence) . .	43	51	61
Lamps now proposed (rated at 240 v.)	30-watt	40-watt	60
Lumens obtained . . .	170	236	362
Lamp renewal cost (pence) . .	6·2	6·2	6·2
Energy cost (pence) . . .	26·3	35·1	52·7
Total cost (pence) .	32·5	41·3	58·9

over half (actually 58 per cent.). There are, however, ce limits to the amount of under-running which is practical any given case. Thus when the life calculated to result such a practice approaches a very high figure, it may be f that vibration or mechanical mishaps will cut it short b all the anticipated length has been realised. It might al thought that there would be a limitation owing to the c becoming unduly red, but even with so great a reduction as suggested above (250 volts to 220 volts) the efficiency is about twice that of a carbon-filament lamp, so that t should not be any serious objection on this score.

Dear Energy Cases.—The other case mentioned above possible application of these principles—in which energ expensive compared with lamps—is rendered difficult by fact that it occurs largely with small consumers and in isol situations, *i.e.*, where technical skill and advice are harde and where the amount of the individual saving would imum. It is too much to expect that the local de

who probably also sells cycles and gramophone records, should advise as to the best life to use, nor is he himself likely to welcome any suggestion which will mean the carrying of additional stock. On the other hand, it is clearly wrong to run a lamp at its normal rating, designed for 4*d*. energy, in a place where each unit of energy costs half as much as the lamp, and where a single lamp in the course of its normal life is responsible for consuming several pounds worth of energy.

Something can be done by localising the problem, since in any one area (particularly a country district) there will not be many, and probably only one energy price to be considered. Dealers in a dear energy district could be advised to stock largely the smaller lamps, and these should be lamps rated for a slightly lower voltage than the actual pressure of supply. Moreover, in considering the saving which might accrue, it would be necessary to consider the group rather than the individual, and to visualise instead of a single large consumer a single large village or small town.

Another difficulty which arises here is that if the price ratio is not greatly in excess of the appropriate figure, whilst it will pay to over-run to some extent, this will change the sizes of the illumination units, and unless the lighting can be redistributed there is likely to be an increase in cost, although, of course, with a still greater increase in illumination. On the other hand, if the energy price is very greatly in excess, the over-running can be on a larger scale, so that smaller-sized lamps can be substituted for the bigger ones originally employed. But owing to the lower inherent efficiencies of the smaller lamps, the saving which results is not appreciable unless the energy price is quite unusually high. It is therefore doubtful, except where energy costs more than 1*s*. a unit, whether over-running is practicable, unless the consumer is willing to have a change in the size of his illumination units, or unless the lamp manufacturers are prepared to earmark some of their " borderline " lamps for use on such circuits.

Other Considerations.—One point should be noticed which applies to either the under-running or the over-running situations. In many cases the actual voltage which reaches the consumer is not the nominal (declared) one, and often it fluctuates over a considerable range. It need hardly be said that the figure employed as the working voltage in the above calculations should be the actual average pressure on the lamps

where this can be ascertained. Moreover, where there are considerable fluctuations these will no doubt lessen the life of the lamp whatever the method of running. At the same time, there seems to be no reason why the consumer should not apply the above principles and get the best value he can under the circumstances, even though these circumstances are far from ideal.

For the most part, only initial conditions have been considered. As a lamp is used its filament deteriorates to some extent, causing in the first place a loss in candle-power, and secondly a loss in efficiency (since the filament is virtually smaller and therefore under-run). This deterioration is far less than it was with the older type of lamps, in which the life was frequently defined not as its life up to breaking point, but that up to when it gave 80 per cent. of its initial candle-power. With the modern tungsten lamp it is rare for even an individual lamp to last until it becomes uneconomical on account of diminished efficiency, and the mean efficiency of a modern lamp should be over 90 per cent. of the initial efficiency.

It is difficult to determine exactly what effect deterioration should have upon the economic calculation, without further data as to how the deterioration affects the values of the indices. It is safe to say, however, that in comparison with the wide range of the other variables entering into the problem, its effect on the economic position will be slight, and its chief result is a lowering of the mean size of the illumination unit.

When several energy prices rule—as when the first portion of the consumption is charged for at a higher rate—the economic choice should not be affected, since a change in efficiency will not affect the time distribution of the consumption. In such a case it is only necessary to employ in the formula the mean price paid per unit of lighting energy over the period in question.

Need for Reform.—The purpose of the present chapter has been not merely to enable the individual lamp user to get the most light for his money, but also to urge the need for a more extensive and conscious application of electric lighting economics, and greater assistance thereto from the lamp manufacturers. The vital connection between rating and economy, and their sensitive dependence upon pressure, are at last being recognised, and Mr. C. W. Sully, speaking at the World Power Conference on behalf of the Electric Lamp Manu-

facturers Association of Great Britain, said: "It is therefore extremely important that lamps should be operated at their rated voltage. . . . It is equally necessary that supply voltages should be as definite and constant as possible, in order that lamp manufacturers may rate their lamps at the most economical efficiency."

But the problem is regarded too statically, and of the many recent attempts at educating the public in the proper use of lamps there has hardly been one pointing out that the correct rating is not something fixed by nature or the lamp manufacturers, but should depend on circumstances, and, in particular, upon the energy price and the lamp size. To cite a particular case, until recently the author was paying 7*d*. a unit for all his lighting energy, whilst blocks of flats near by were buying their energy at a uniform price of 1*d*. Lamps which were economical for the one circuit would be quite unsuitable for the other, yet the same shop served them both, and there was nothing on the lamps or their wrappers to show that they were not suitable for the same voltage under all possible circumstances. In a similar way 20-watt and 60-watt lamps for the same circuit should be designed to have quite different lengths of life.

The energy price is admittedly the most difficult item in the problem, and the above suggestions make no pretence of exhausting the possibilities. But, however difficult, there can be no denying the importance of the problem, in view of the fact that the energy cost may frequently represent 90 per cent. or more of the total cost of the illumination, and that the energy price may vary as much as from 1*d*. to 1*s*. a unit. In the absence of any more positive action or assistance for the consumer, it would appear that the least that could be done would be to stamp on the lamp or wrapper the particular energy price as well as the voltage for which the rating was determined.

Illumination and Output.—Although this chapter has so far dealt exclusively with the economics of lamp choice, and in so doing is in line with the other chapters in this section of the book, there is an entirely different economic question in connection with illumination which should be mentioned. In talking of the efficiency or economy of a lamp, one is concerned with the ratio :—output in luminous intensity (for a number of hours) divided by input in kW hours or in £ s. d. But as the illumination is only a means to the end of getting certain work

or recreation accomplished, it is possible to consider a further stage in the process and examine the economy not of the lamp, but of the illumination. The " output " of this process can be regarded as the amount of work accomplished or satisfaction achieved, and the " input " is the degree of illumination employed or the cost of this illumination in relation to the cost of the other factors involved. (There is, of course, a stage in between these two, namely, the process by which luminosity in the lamp becomes illumination of the object.)

There have been a number of experiments, notably in America, to measure the increase of output and decrease of the errors, accidents, etc., resulting from increased illumination in various industrial processes. It is only to be expected that the finer the work the higher the degree of illumination required, but even in the relatively large-scale operations of heavy engineering, there is reason to believe that the illumination generally employed is too low, and that an increase in it will result in less errors and accidents, even though the increase in output cannot always be measured. As regards fine work involving delicate hand operations, a very notable increase in speed can frequently be obtained by increasing the degree of illumination during those hours in which daylight is not available. Some of these operations, such as letter-sorting, typesetting, etc., lend themselves well to actual measurement of output, and one of the cases recently investigated in this country is referred to below. The amount of speed increase obtainable with increased illumination naturally depends chiefly upon the proportions in which the normal slowness is due to delays in perception on the one hand and to physical inertia or time lag on the other. But there seems reason to believe that in almost all cases the illumination as planned by the normal uninformed employer is too low, and an increase would pay him, even taking the shortest possible view of the case.

The only experiments which it is proposed to mention here are those carried out in 1926 under the auspices of the Medical Research Council and the Department of Scientific and Industrial Research on " The Relation between Illumination and Efficiency in Fine Work (Typesetting by Hand).* Experi-

* Joint Report of the Industrial Fatigue Research Board and the Illumination Research Committee, 1926. H.M.S. Stationery Office, price 6*d*. Since the above section was written a very good summary of this work has been given in a paper by H. C. Weston, *The Illuminating Engineer*, August, 1927.

ments were made with five different values of artificial illumination and also with daylight, and records were made of the average hourly output, the number of errors and the number of "turned letters" which resulted. The output (measured in the number of letters of average width set per hour) and the number of errors per cent. of the average output, are plotted in Fig. 12 to a base of illumination (even scale), and it will be

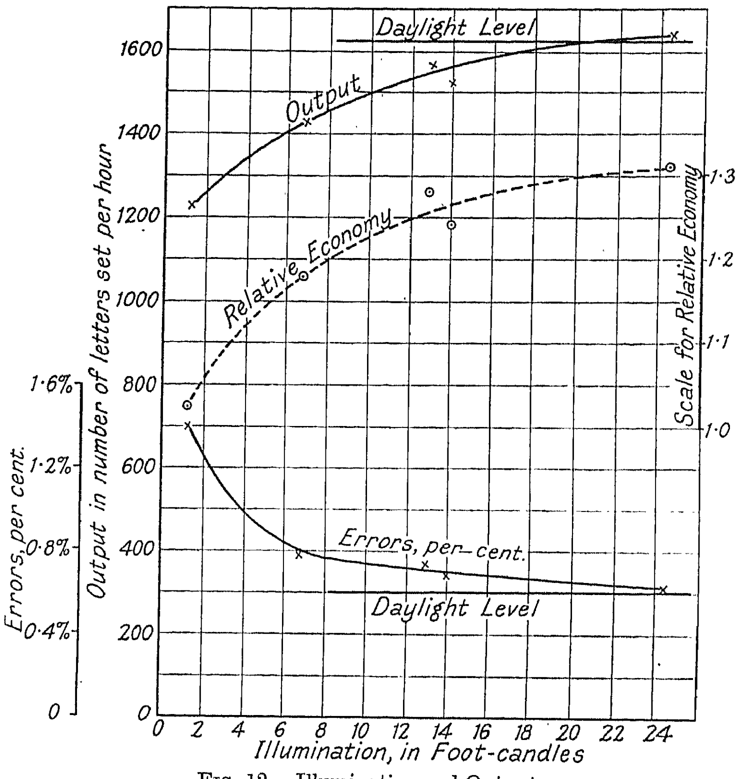


FIG. 12.—Illumination and Output.

seen that the actual points are indicated by crosses, whilst horizontal lines indicate the values achieved with daylight. It was found that the output increased and the errors decreased steadily with increase of illumination, but at a slower rate as higher values were reached. (Of the two points close together, one represents the use of an artificial daylight unit, and, whilst slightly smaller in magnitude, resulted in a slightly greater output.)

When an illumination of the order 20 to 25 foot-candles was reached, the output and errors approached the values obtained with daylight, and the results, although not great in number, are so consistent among themselves that they appear to justify the definite conclusion that there is an "optimum" value of illumination for hand-composing of about this value. It will be noted that the magnitude of this is far less than the value of the illumination due to daylight which, although it fluctuated considerably when the daylight test was made, was rarely less than 50 foot-candles, and rose as high as 500. But with artificial light, it is difficult to increase the illumination much above the figures used in the test without running the risk of complaints of glare, unless very great care is taken with the distribution. In any case, the value of 20 foot-candles is, so far as can be ascertained, a very much higher illumination than is usually found in printing offices at present.

In the Report from which the above figures are taken the results are plotted with a logarithmic base scale, and the output then follows a straight line, so that in order to gain any given arithmetic increase in output requires a particular geometric increase in the illumination. In the case given, doubling the illumination added to the output by 320 letters per hour, *i.e.*, about one-fifth of the hourly (daylight) output. Even if this law were followed indefinitely, so that still further increases of illumination gained gave greater and greater output, there would still be an economic limit to the increase which was worth while, since the cost of doubling the illumination would get steadily greater the bigger this became, whilst the value of an increase of 320 in the output would always be the same. In attempting to show this point, a third curve has been drawn dotted on Fig. 12, and labelled "relative economy." The data employed is extremely tentative, and must be regarded as an illustration of method only. The cost of the illumination is taken as 5*s*. per 1,000 hours per foot-candle falling on the working plane, the cost of the labour, plant and overhead expenses is taken as 3*s*. per hour, and the value of the work done is assessed at 1*s*. per 400 letters set.

The relative economy was defined in Chapter V. as the ratio $\frac{\text{Value of Output}}{\text{Total Cost of Input}}$, the values and costs referring to the service in question. In this case it becomes—value of work done divided by cost of labour, etc., plus cost of illumination; and

putting in the rates mentioned above, the dotted curve shown is obtained. Comparing this with the output curve it will be seen that it rises more steeply at first, but becomes flatter later, and even if the output curve continued to rise with additional illumination it is evident that the *economy* of the process would shortly reach a maximum and commence to decline.

Additional Worked Examples

2. Two types of lamp, each nominally rated to take 60 watts, give the same candle-power. One type costs 2*s*., has a life of 1,000 hours, and consumes exactly 60 watts. The other type costs only 1*s*., but is found on test to have a mean life of 750 hours, and to consume 65 watts. For what energy price will the two be economically equal ?

In 3,000 burning hours, three of first type lamps would be required, costing	6*s*.
In the same time, four of second type lamps would be required, costing	4*s*.
Difference in lamp cost=2*s*.	

In this period the second type of lamp will have consumed $5 \times 3 = 15$ kW hours more than the first.

Hence, energy price at which they will be economically equal is $\frac{2s.}{15} = 1{\cdot}6d.$, and for any price higher than this the first type of lamp will be preferable.

3. Two makes of 40-watt vacuum lamp have the following characteristics :

Lamp (*a*) costs 2*s*. 6*d*., has 1,000 hours' life, and gives 8·1 lumens per watt.

Lamp (*b*) costs 1*s*. 4*d*., has 800 hours' life, and gives 7·5 lumens per watt.

Determine which will be the more economical, first with energy at $1\frac{1}{2}d$. per unit, and secondly with energy at 5*d*.

BASIS OF COMPARISON—4,000 HOURS BURNING

Alternative.		(*a*)	(*b*)
Lumens given at 40 watts		324	300
Number of lamps required in 4,000 hours		4	5
Cost of ditto		10*s*. 0*d*.	6*s*. 8*d*.
Cost of energy at $1\frac{1}{2}d$.		20*s*. 0*d*.	20*s*. 0*d*.
,, ,, 5*d*.		66*s*. 8*d*.	66*s*. 8*d*.
Total cost (shillings) per 10^6 lumen hours	at $1\frac{1}{2}d$.	23·15	22·22
	at 5*d*.	59·16	61·11

It will be seen that the cheaper lamp (alternative (*b*)) costs less per lumen-hour than (*a*), when energy is at $1\frac{1}{2}d$. per unit, to the extent of nearly a shilling per million lumen-hours ; but it costs more than (*a*) when energy is at 5*d*. to the extent of nearly 2*s*. per million lumen-hours. It will be noted also that the use of lamp (*b*) involves a smaller illumination unit, which may be an advantage, and the decision must, therefore, depend also upon which size is preferred.

PART III

SUPPLY PROBLEMS

CHAPTER XI

TARIFFS

Underlying Considerations.—Of the making of tariffs there is no end, and when they are discussed at the Institution or in the technical Press the argument aroused is so considerable that it has been suggested there ought to be a "close season" for tariff discussions. In this country alone there are some dozens of different types of tariff in use, and many more have been suggested or tried. The trouble largely is that not only is it difficult to frame a tariff to satisfy a given set of requirements, but it is difficult to agree even as to the requirements to be satisfied. Broadly speaking, there are four considerations which may underly the fixing of a tariff, and of these only the first is a purely economic one, the others being based upon questions of business or expediency. They are not mutually exclusive, and as far as possible all four should have a place in the tariff chosen. These are :—

(1) Cost of production.
(2) Service rendered.
(3) Ability (or willingness) to pay.
(4) Acceptability to public.

The first of these considerations is usually the most important, and the majority of tariffs are based primarily on this, modified to a greater or less extent by the other three. It follows the strictly economic aim of making every consumer pay his fair share of the total cost, which may be defined as all the expenses incurred in giving him his supply, and which would not be incurred if he (and his like) were not there. (It is assumed either that he is a sufficiently big consumer to make an appreciable difference, or that he is grouped with others of the same sort in order to estimate the costs incurred.)

The other three considerations have been spoken of as based on questions of policy rather than strict economics, but the distinction is difficult to maintain. If "cost of production" is regarded, not statically for the undertaking as it exists at present, but with an eye to the future, it must concern itself not only with present costs, but with the costs which would be

entailed if certain likely developments took place. A tariff could therefore be said to be "economic" though charging less than present cost of production to certain consumers, if it can be anticipated that the business ultimately resulting therefrom will finally pay for all the costs incurred. Nevertheless, although the distinction is not a rigid one, it will be convenient to regard the three remaining considerations as being questions of policy as distinct from immediate strict economics.

As regards the second consideration, namely, value of service rendered, it is clear that there is more obvious human satisfaction to be obtained from one kilowatt-hour spent in lighting than from the same spent on heating—the former will light a large room on the darkest winter night for ten hours, whilst the latter will hardly heat the same room for more than twenty minutes or half an hour. In this respect illumination is a little like wireless broadcasting with crystal reception—the sensitivity of the receiving apparatus (in this case the eye) is so enormous that only the tiniest fraction of the electrical energy consumed need reach its goal. Thus electricity* for lighting is very frequently priced higher than that for any other purpose, quite apart from and in addition to the differences made for other reasons. In general, it may be said that the differences in the effective yield of a unit of electricity, in light, heat and power, call for different charges so as to compete with other available forms of energy.

As regards the third point, ability or willingness to pay, it is evident that a high-class residential district can be got to pay rather more for its electricity than a poorer district or a manufacturing area, quite apart from the purpose to which it is put. Here again, the lighting energy is likely to be priced higher, since electric lighting, whatever it costs, is often a virtual necessity, not only because of its obvious merits but also for its "kudos," *i.e.*, social or advertising value. For the most part, however, it is difficult to apply either consideration (2) or (3) by themselves, and whilst from the economic point of view the former goes with and is measured by the latter, from the human point of view they are frequently in opposition. The actual physical need for electricity, with its absence of fumes, etc., is greatest in the dark and overcrowded

* The somewhat loose term "electricity" has been used here and elsewhere in preference to the more rigid "electrical energy" in order to emphasise the fact [illegible] supply is a service rather than a mere sale of kWh, its cost [illegible] many things besides the amount of energy taken.

rooms of those who are at present least able to afford it; whereas economically the service rendered can only be taken as the gratitude of the person served, expressed in the £ s. d. which he is willing to pay, *i.e.*, by "value" in the economic sense rather than "worth."

Hence considerations (2) and (3) can hardly be separated in a purely business concern, since the "service rendered" has to be measured not by intrinsic worth, but by effective economic demand—*i.e.*, "willingness to pay," which is largely a question of the cost of the available alternatives. But in the future these two may have to be considered quite separately, when electricity supply comes to be regarded as a universal national or international necessity, like a pure water supply or cheap postal service. It may then be thought worth while for reasons of national policy, such as the encouragement of rural areas, to run some sections permanently at a loss; just as the G.P.O. carries single letters to outlying villages in Scotland or India at a loss, which is more than made up by the enormous volume and simplicity of a system giving identical postal facilities to every member of the British Commonwealth.

The last of these considerations is a psychological one—the acceptability of the tariff to the consuming public. From this point of view the tariff should appear reasonable and fair, making no charges which seem devoid of economic justification, and no differentiation between apparently identical consumers. Above all, it should be as simple as possible, with the minimum of technicalities, so that the consumer may know just how he stands, and, if possible, may himself be able to check the bill from the meter readings.

The purpose of this chapter is not to give any full account of electricity supply tariffs, but merely to indicate the lines which these take, and some of the reasons and motives underlying them. Specific economic differences such as those due to load and diversity factor, power factor, etc., are dealt with in subsequent chapters, together with some of the tariffs which result.

Simple Tariff and its Limitations.—The simplest possible tariff is that in which electrical work or energy (the two mean the same thing) is sold at a fixed rate per kWh (2·65 million foot-pounds), just as potential energy in the form of gas is frequently sold at a fixed rate per "therm" (77·7 million foot-pounds). This was the original statutory method of charge,

B.O.T.

the kilowatt-hour being the Board of Trade "unit" for charging purposes. (Where no ambiguity is possible the word "unit" has been very generally used in this book as an abbreviation for kilowatt-hour.) It still forms the basis of most existing tariffs, but by itself it is far from representing the true costs of supply as between one consumer and another. When the various classes of consumer in any given area are examined it will be found that there are four main differences which affect the cost of supply per unit:—

(1) Differences in the magnitude of the consumption. It is cheaper to generate a big quantity than a small, and less expense is usually involved in supplying one large consumer than many small ones. This, which is a feature of practically all forms of production and is usually met by giving discounts for quantities, is a less important item in electricity supply than the other differences mentioned below.

(2) Differences in the time characteristics of the consumption. These differences, which are expressed by means of the load and diversity factors, are dealt with in detail in the chapter which follows. Some few types of consumption require an almost steady and unvarying current, for purposes such as battery charging and electro-chemistry, and this is naturally the cheapest to supply. The remaining consumption (usually the great majority), which fluctuates with the time of day or year, can be roughly split up into two groups—that which occurs regularly each day at the same time as the peak load due to other consumers (this is the most expensive to supply and is usually the lighting portion), and that which is taken during non-peak periods and so presents a good diversity factor relative to other consumption.

(3) Differences in the electrical characteristics of the consumption (in an A.C. case), which can be expressed in the form of the power factor or the phase relationship between current and pressure. This is dealt with in detail in Chapters XIII. and XIV.

(4) Differences in the electrical characteristics of the supply. Since transmission almost invariably takes the form of high tension alternating current, it follows that it is normally cheaper to supply A.C. than D.C. and high tension rather than low. In comparing consumers who are not in close proximity there are also differences of location which can be put into this group.

These various differences, and particularly the last three,

form the economic subject-matter for most of the remainder of the book. As regards the various tariffs which have been devised to meet them, these are also touched upon in the chapters referred to, although no attempt is made at an exhaustive statement of the numerous ones in use at the present day, still less at a detailed criticism and comparison. As already explained, the present chapter deals with general principles only, and endeavours to state the main items going to make up the problem and to indicate the possible lines of solution.

Taking a general view, it will be evident that the first and the last are the easiest to deal with. The first can be met, where desired, by a discount for quantity, and the last refers to factors which do not fluctuate with the consumption, but are fixed at the time of installation. Whatever the other features of the tariff, it is a simple matter to allow a reduction or rebate to consumers taking at a high tension, or taking A.C. rather than D.C., or living nearer to the sub-station. The other differences are harder to cater for, since they are varying all the time.

Modifications of Simple Tariff.—The following are the principal modifications of the simple energy tariff or flat rate per unit :—

(1) Discounts for quantity.
(2) Special terms to special consumers or types of consumer ; and/or several different rates co-existing when electricity is used for several different purposes.
(3) Modifications to suit metered load factor.
(4) Modifications to suit estimated load factor.
(5) Modifications to suit time of day or year.
(6) Modifications to suit power factor.
(7) Modifications to suit change in fuel prices.

In addition to the above, there are cases in which the kWh are not metered at all, the whole charge resting on some estimate or measurement of the maximum power demand and the probable amount of consumption. The power may be kept within the contract figures by means of a current limiter (flicker instrument).

It will be seen that all of these except the second confine themselves to the strictly economic motive of making the consumer pay his fair and exact share of the cost of production.

Taking them in the order given, the first modification needs no explanation and is generally not of great importance.

Modification (2), the use of different rates for different purposes, is particularly applicable to domestic consumption, where the electricity used for lighting is usually priced at about three times that used for heating, cooking, etc., and this in its turn is more expensive than the electricity used in industry. In the case of lighting, practically every one of the factors already mentioned (poorer load factor, worse time of day, greater service rendered and ability to pay) help to account for the higher price. Unfortunately this method of charge usually necessitates separate circuits and wiring for the different purposes for which the electricity is used. The modern tendency is therefore in the direction of abolishing this distinction between "lighting" and "heating," so as to encourage *all* domestic consumers to take as varied a load as possible.

As an example of the third modification, the maximum demand system using two metering devices may be cited. In its usual form one meter or attachment shows the biggest current or power taken (for an appreciable period, such as half an hour) during the quarter, month or other metering period, whilst the main meter registers the actual energy consumed. The biggest current taken, when multiplied by the line voltage, gives the maximum demand in watts, and this may be paid for by a standing charge of so many pounds per annum per kW of demand, whilst the energy consumption is then charged at a low rate of some fraction of a penny per kWh. This method is largely employed for big power consumers.

As regards the fourth modification, instead of metering the demand in addition to the energy consumption, reliance may be made upon estimates of the former (or occasionally of the latter). Thus the probable maximum demand may be calculated from the capacity of the plant installed, or (with a domestic consumer) the number of points, the size or rateable value of the house, or other data. In this way the double tariff principle can be applied to a small consumer without the necessity for two sets of meters and readings. This system may also have the advantage of being more easily explained to the consumer, and it resembles the telephone tariff of an annual rent plus a low *pro rata* charge. On the other hand, when based on plant capacity or number of points (rather than

floor space or rateable value), it may tend to discourage the introduction of additional or more varied plant, and so may have a bad effect on the diversity factor.

Modification (5) is also comparable to one type of telephone charge—that for trunk calls, which are cheaper at some times of the day than at others. Unfortunately in the case of electricity supply it necessitates rather complicated metering, in order to change the rate of registration twice each day, and it is not very much used in this country. A variation of this, which has been tried on a small scale, is to have two separate circuits with *alternative* switching. The second circuit may contain heaters of the storage type and other apparatus which can be connected during the "off" periods.

The sixth modification, to suit differences in power factor, can be fairly simply combined with a maximum demand system by making the fixed charge depend upon the power factor at the time of demand. This is dealt with in detail in Chapter XIII.

The last of these modifications concerns fuel price. When a contract is made with a large consumer and covering several years ahead, it is usual to insert a coal clause. This may provide that the price per unit shall advance or recede by a definite amount for each shilling in the ton rise or fall in the price of the particular grade of coal employed. In a two-part tariff it is, of course, only the energy charge which is varied in this way. Occasionally there is also a wages clause, though there does not seem much reason for this in view of the small amount of labour directly employed. A typical two-part tariff containing a coal clause is given at the end of the next chapter (see p. 223).

CHAPTER XII

LOAD AND DIVERSITY FACTOR: TWO-PART TARIFF

Load Factor: Definition.—The load factor can be defined as the actual energy consumption divided by the consumption which would have occurred had the maximum power been taken all the time. It may refer to any particular load or part of a load, and it may be taken over any specified period such as a day or a year. When the apparatus concerned takes its full power all the time it is connected, then the load factor is merely the time of connection divided by the total time—thus plant which takes, say, 100 kW whenever it is connected, and which is in circuit eight hours a day, will have a load factor of one-third, or $33\frac{1}{3}$ per cent., whilst if it is in circuit six hours a day the load factor will be 25 per cent. Unfortunately, most apparatus not only is not connected continuously, but does not take its full current all the time it is connected, and this results in a still further reduction of load factor. The maximum power or current taken by a consumer, a cable or a circuit is called its "maximum demand," and the load factor can therefore be defined as

$$\frac{\text{actual consumption over a period}}{\text{maximum demand} \times \text{length of period}},$$

the numerator and denominator being expressed in the same units.

There is something resembling load factor in almost all forms of production: machinery, ships, lodgings, taxicabs, in fact, practically all capital goods are necessarily idle during periods of their existence, and during these periods they cost money in interest, depreciation and standby charges, which has to be paid for during the working period. The longer this idle time is per annum, in the case of any given plant, the greater will be the capital charges then incurred, and the less the earning time available. Hence the improvement of load factor is at all times a thing to be aimed at, although in many branches of production the economic limit is soon reached beyond which further improvement is not worth its cost. Thus in a factory

employing machine plant and working eight hours a day, the actual capital charges per item of product will necessarily be more than they would be if an overtime or night shift be worked: but unless the machinery is very expensive the saving resulting from overtime would be more than balanced by the higher cost and lower efficiency of labour during these extra hours, quite apart from the human inconvenience involved. In most factories, therefore, overtime is rather a temporary expedient than a permanent economic advantage, except as regards particular and highly expensive machines. With electricity supply, as will be seen below, the balance between machinery and labour is altogether different, and an improvement of load factor will almost always produce a very great cheapening of supply.

Magnitude and Effect.—It is easy to see that the load factor of an individual consumer will generally be low. Thus a power consumer working a nine-hour shift, which, allowing for short time, holidays, repairs, etc., may be estimated to extend over 300 days per annum, even if his mean rate of consumption during the shift were 75 per cent. of his maximum consumption, will have a load factor of only $\frac{9 \times 300 \times 0{\cdot}75}{24 \times 365} = 0{\cdot}231$, *i.e.*, 23·1 per cent. A domestic consumer using electricity for lighting is likely to be even worse. Thus, suppose that he has twenty equal lighting points, of which at night time an average of four are employed continuously and a maximum of seven at any one time.* The mean daily hours of burning might be estimated at the time of the equinox, say, from 6.30 to 11 p.m., *i.e.*, four and a half hours, but the effect of the Daylight Saving Act has been to reduce this somewhat. Taking a mean of four hours per evening, his load factor would be only $\frac{4 \times 4}{24 \times 7} = 0{\cdot}095$, *i.e.*, 9·5 per cent. Fortunately these low individual load factors are redeemed to a considerable extent by the diversity factor discussed below.

With regard to the effect of load factor upon the costs of supply, if A is the cost per annum of a station whether generat-

* The maximum, if metered, will probably be read from an instrument taking twenty to thirty minutes to reach its final position. Thus accidental overloads or temporary switching on of extra lights will not be registered.

ing or not, and B is the cost per unit generated, then for an output of u units per annum the total cost will be $A + Bu$, or the cost per unit $\frac{A}{u} + B$. If the load factor be improved n times, the cost per annum will be $A + Bun$ for an output of un, *i.e.*, the cost per unit will be $\frac{A}{un} + B$. Thus the effect upon the cost per unit of improving the load factor n times is that the working cost remains the same, but the standing cost is reduced to $1/n$th.

The gain will therefore be dependent not only on the magnitude of n, but upon the original size of the standing charge in relation to the working charge. If the original standing charge was four times the working charge, the original cost per unit could be represented by the figure 100, of which 20 would be working and 80 standing. If the load factor were then improved four times, *e.g.*, from 10 per cent. to 40 per cent., or from 25 per cent. to unity, the new cost per unit would be only 40 instead of 100, of which 20 (as before) would be working and 20 standing. Thus the standing charge would be reduced to one-quarter, and the total charge would be 40 per cent. of its original value.

As will be seen later on in the chapter, the standing charge in electricity supply is the chief item of cost, and the effect of load factor is therefore paramount. When load factor and size of station go together the difference in costs between a large system and a small one becomes enormous, as can be seen in the comparison between London and Chicago on p. 279. Even within our own country and with no other changes, an improvement in load factor may be said to be the shortest cut in the direction of price reduction, since (and this is a fact which is frequently overlooked) the chief expense in the production of electricity lies not in quarrying or otherwise obtaining the potential energy, but in converting it into a useful form.

Many imaginative writers, in planning their Utopias of the future, have suggested that if we could tap some cheap inexhaustible source of energy such as that of the atom, all our troubles, at least as regards cheap power and labour saving, would be at an end. But if the plant required to convert this energy into a utilisable form were as expensive as our present plant for converting coal energy into electricity (which it prob-

ably would be), we should be little better off nationally, to recompense us for the loss of one of our largest industries and the ruin of those engaged in it. Taking the latest year for which figures are available (1924), the fuel accounted for less than 20 per cent. of the total cost of supply, and it is a striking fact that *even if collieries produced coal for nothing and railway companies charged nothing for carrying it, the total reduction in the price per unit which could be effected would be no greater than could be obtained at this very moment by increasing our national load factor from* 30 *to* 40 *per cent.**

Diversity Factor.—If there were a hundred consumers connected to a station all with identical loads in every respect, the energy consumption would be 100 times the individual one and the maximum demand would also be 100 times, so that the station load factor would be the same as each consumer's load factor. But actually the consumers vary, so that whilst the station consumption is always the sum of all the individual consumptions, the station demand is less than the sum of the individual demands. For even if two consumers had the same loads and load factors they would probably not take their maximum currents at the same instant, and, in fact, their consumptions might not even overlap, so that the station demand would be far less than double the individual one. Owing to this fact of different consumers taking their maxima at different times of the day and year, the maximum demand on the station is always less than the sum of the consumer's maxima, and hence the station load factor is always better than the average consumer's load factor. In an extreme case it would even be possible for two consumers, each with a load factor of 50 per cent., to combine to give a station load factor of 100 per cent.

The sum of the consumers' maxima divided by the actual maximum coming on the station is called the diversity factor. Since the maximum demand forms the denominator of the load factor fraction (see p. 201), and since the numerator (actual consumption) is necessarily the same for both consumer and station, it follows that the station load factor will be the mean consumer's load factor multiplied by the diversity factor.

* It is assumed that the improvement in load factor is obtained by an increase of one-third in the consumption without any increase in the maximum demand. Such a load could be handled by our present equipment as it stands, and would lower the mean overall price per unit in the country by the quantity stated, namely, the total amount which at present goes in fuel.

(The mean, of course, refers to a mean of consumption, not of consumers.) Diversity factor can be expressed not only as between consumer and station, but also can refer to the conditions before and after any point at which a number of circuits or cables meet. It will be seen that diversity factor cannot be less than unity, whilst load factor, like power factor, cannot be greater than unity.

The importance of diversity factor in lowering the maximum demand on a station and so improving the load factor is very great. A lighting consumer taking energy for a few hours and at a particular time each day has both a bad load factor and comparatively little diversity factor with reference to other lighting consumers; but even here the item is considerable, since some users start earlier, others finish later, and very few synchronise exactly their biggest demand. Power and heating users will generally have better individual load factors, and they also vary considerably amongst themselves—bakeries, restaurants and domestic heating consumers taking their biggest loads when factories are not running—whilst the diversity factor with reference to the lighting load will, of course, be excellent. Hence the importance to the supply authority of getting consumers not only with good individual load factors, but also using electricity for as many different purposes and at as many different times of day as possible.

Degree of Capitalisation.*—The outstanding economic feature of electricity supply may be said to be the high ratio of standing charges to working expenses, due to the expensiveness and comparatively short life of the plant employed, intensified by the difficulty or impossibility of storing the energy generated. In order to obtain a clear conception of the magnitude and significance of this ratio it will be instructive to regard electricity supply first as a commodity compared with other manufactures, and second, as a service compared with other services.

In contrasting the generation and distribution of electricity with the manufacture and delivery of other commodities, a useful basis of comparison will be with the engineering industry, itself a highly capitalised one. According to the last available information, the *Census of Production*, 1907, the total engineering firms of the country produced a net output in that year of the value of £99,000,000, of which almost

* This analysis, with diagrams, is reprinted from *World Power*, January, 1927.

exactly 50 per cent. went in payment of material and 30 per cent. in wages.* Hence the whole of the overhead and capital charges absorbed only 20 per cent. of the total, and if this sum represented an average of 10 per cent. on the invested capital, the total investment would be just double the total annual receipts. In electricity supply the invested capital is nearer six times the total annual receipts, and as a consequence something approaching half these receipts are absorbed in capital charges alone as compared with less than 20 per cent. in the engineering industry.

In contrasting electricity with other sorts of supply such as gas, there are fortunately ample up-to-date figures available. Taking the five largest municipalities which own both electricity and gas undertakings,† the following totals are obtained, reckoned in each case as a percentage of the total annual receipts. As a contrast in the opposite direction to that of gas (*i.e.*, of a service in which practically the whole of the expense is overhead), the figures for the water supply, also municipal, are given in the table for the same five towns :—

	Electricity.	Gas.	Water.
Total capital outlay (as percentage of annual receipts)	590	250	1,330
Excess of receipts over working expenses (as percentage of annual receipts) (available for interest, debt redemption, etc.) ‡ .	47	9·3	57

It will be seen that of the total annual receipts from the five large electricity undertakings 47 per cent. was absorbed in

* The figures for wages are not given in the Census, but the total number of employees is given, and for the purpose of this estimate this total is multiplied by the average wage in September, 1906, as given in the Board of Trade Inquiry on "Earnings and Hours in Metal Engineering and Shipbuilding Trades, 1906 (Cd. 5814)."

† Manchester, Birmingham, Leeds, Edinburgh, and Bradford. Figures from the *Municipal Year Book*, 1926.

‡ It will be noted that this includes no profit item, but on the other hand it covers a high depreciation allowance in the form of the comparatively rapid debt redemption which is statutorily imposed. The Electricity Commissioners' Return, discussed later, shows no substantial difference between Municipal and Company undertakings as regards the percentage of excess, so that the diagram shown may be regarded as representative of all types of electricity undertaking.

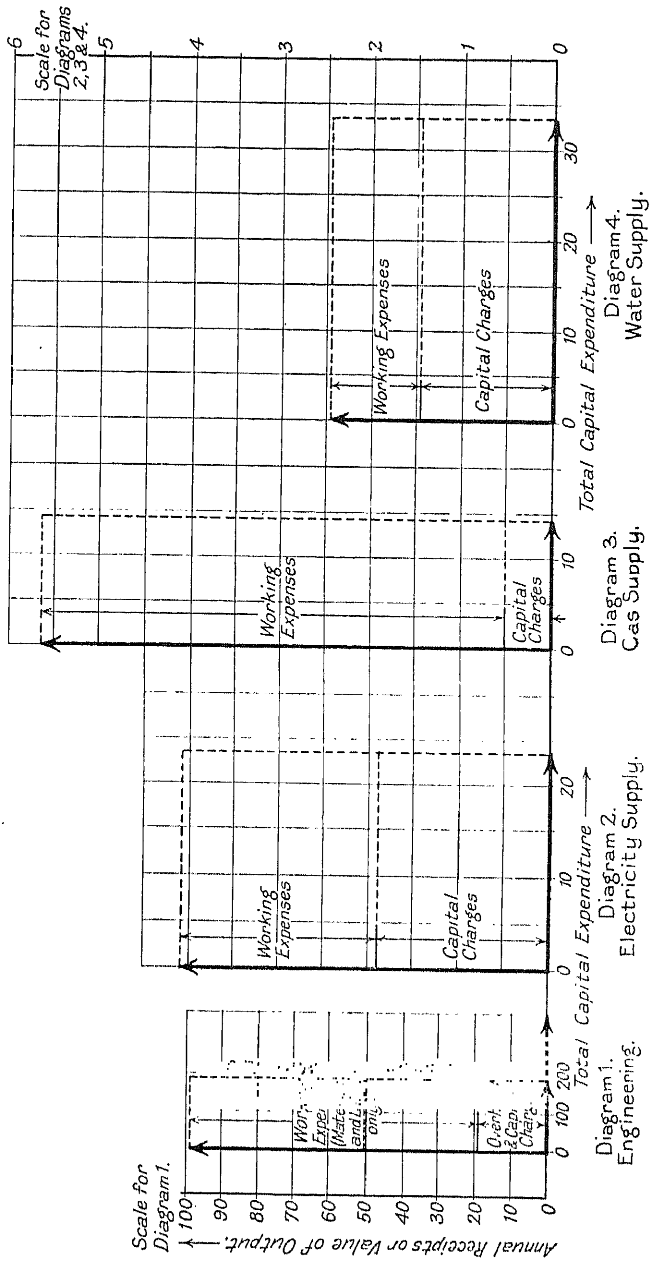


FIG. 13.—Degree of Capitalisation. (All numbers represent 10^6 £.)

capital charges, as compared with 9·3 per cent. in the gas undertakings.

In the diagrams, Fig. 13, the arrowed heights represent annual receipts or value created, whilst the base measurements represent capital expenditure to one-tenth the scale. In the first two diagrams, engineering for the whole country in 1907 is compared with electricity supply in 1924–25 for the five towns mentioned. The latter is drawn to twenty-five times the scale, thus making the two heights practically the same, and it will be seen how differently made up this height is in the two cases. (It is interesting to note that since the working expense of electricity supply is very largely fuel, *i.e.*, material, the proportion of the year's product which goes in materials is not so very different in the two cases.)

The capital invested in engineering works is not known, so that the base line of Diagram 1 is of uncertain length. The full portion of the line is drawn on the assumption that the total standing charges represent 10 per cent. on the total capital expenditure. This makes the dotted rectangle containing the words " overhead and capital charges " into a square.

Taking the second, third, and fourth diagrams, these represent respectively the electricity, gas, and water supply for the five towns mentioned, and all drawn to the same scale. It will be seen at once that the sale of electricity brings in only about three-quarters of the receipts from the sale of gas although the capital invested is nearly double. Water supply represents a further extreme—still heavier investments and still smaller receipts and working expenses.

Another point which is brought out in the diagrams is the high proportion of the depreciation and obsolescence element in the cost of electricity supply. Not only is the plant employed, being elaborate, liable to deteriorate, and, being novel, open to improvement, but the industry itself is in a sense precarious. Mankind has always needed a water supply, and it is humanly certain that he always will; but he managed for a long time without electricity, and might do so again, or it might come to be manufactured in other ways. Thus the dotted rectangle containing the words " capital charges " is nearly square in the case of electricity supply (*i.e.*, nearly 10 per cent. of the capital expenditure), but markedly oblong in the case of gas and water.

The natural result of finding electricity supply (in its economic aspect) occupying a mid position between those of

gas and water, is to look for a method of charging for electricity intermediate between those which have been found suitable for gas and water supplies. And since gas is very generally sold at a fixed price on the metered quantity and water on a fixed rating independent of quantity, it seems reasonable to combine both methods of payment in the charge made for electricity. The two-part tariff is therefore the logical outcome of an economic survey of the supply industry.

Two-Part Tariff.—The suggestion was originally made by the late Dr. John Hopkinson in 1882 that the tariff should be made up of two parts, one to pay for the standing costs of supply and the other to pay for the working costs. The former will then take the form of a quarterly or annual charge based usually on some measurement or estimate of the maximum demand (*i.e.*, biggest rate of consumption), modified possibly by the character of the load and probable diversity factor with other consumers. The second part of the tariff will consist of a comparatively small flat rate per unit, which will usually be based on actual measurement by an integrating meter, but this in turn may be modified by discounts for quantity, etc. Moreover, in either or both portions of the tariff there may be modifications on A.C. circuits on account of power factor. But in spite of all these modifications, and of the many different methods of measurement and tariff now in use, the basic principle of a periodic charge per kW or per kVA for being ready to supply, plus a price per unit for the energy actually supplied, underlies the great majority.

It will be noticed that this supply problem is an inversion of the installation problem dealt with in the second section of this book—instead of being given a fixed set of prices and asked to devise the most economical installation, the installation is fixed and the problem is to determine the most economical prices. The two problems are, however, similar in respect of the principles used in their solution, since in each case the chief thing is to divide the expenses concerned into two groups, standing and working.

The first group must include every item which is a function of time and roughly independent of output—*i.e.*, it must include not only interest, depreciation, and upkeep of the plant, rents, taxes, and management expenses, but also practically the whole of the wages, salaries, and other items classed as running expenses, except fuel. This group

therefore represents the cost of maintaining the equipment ready to meet the biggest demand coming on to it, and is to be met by that portion of the tariff levied on the kW of maximum demand. The second group of costs should include only those items which are proportional to the actual units generated, and it should be met by the energy charge in the tariff.

The fixing of the dividing line between the two groups must be to some extent arbitrary, and in the first case represented below, the whole of the fuel is put in the energy section and all the other expenses are included in the other group. This involves some inaccuracy in both directions—a certain amount of fuel, which may be anything from 5 to 20 per cent. of the whole, is required to keep the plant "banked" ready for action, and on the other hand there are other items, such as oil and water, which are dependent on output; moreover, the percentage assigned for interest and overhead charges includes a certain amount of profit which should perhaps be spread over both standing and running groups. A later example (p. 217), which aims rather at showing manufacturing costs than accounting for selling prices, includes no "profit" item, and in this case 5 per cent. of the fuel, oil, and water has been grouped under the standing charges.

Some mention should be made here of a point of view which differs considerably from the above. Whilst not denying that a compound charge composed of fixed and energy portions must form the basis of any accurate representation of costs, some authorities favour the putting of all possible items (*e.g.*, management, profit, hire of plant during working periods, etc.) into the energy portion of the tariff and leaving as little as possible to be included in the fixed charge. It is also suggested that the fixed charge should not be based so exclusively on load factor as measured by maximum demand, since many of the costs as regards transmission and distribution depend as much upon the space distribution (*i.e.*, density) of the load as upon its time distribution. The data and treatment here employed follow the more traditional assumptions, but it is not suggested that these are in any way final or sacrosant. Any other allocation of costs could be illustrated by means of the diagram shown below, but for the purpose of explanation it is naturally preferable to start with the most familiar and generally accepted hypothesis.

Two-Part Diagram.—Electricity supply is a process in which exact measurement is possible at almost every point, and this, combined with the high degree of publicity, should make it easy to devise an economic tariff acceptable alike to the undertaking and to the consumer. But figures and publicity by no means ensure a satisfactory grasp of the position, and most laymen and even some engineers have a very hazy idea as to the proportions of the various items making up the cost of electricity supply, so that suggestions are often made which a more quantitative conception would show to be absurd. Moreover, even when a really scientific tariff has been devised, it is by no means easy to explain the position to the consumer, or to justify the various charges made at the different points or under particular circumstances.

In order to overcome this difficulty the author has devised a diagram representing the two-part tariff and the figures on which it is based in a semi-pictorial fashion so that they can be at once visualised, and yet in such a manner that quantitative exactness is not sacrificed. The diagram endeavours not only to link up and explain the price charged for energy in terms of the various cost items, but also to show what the price might have been had the energy been taken at a different load factor or in another form or place.

The basis of the charge for electricity is taken to be the simple two-part tariff of so much per annum per kW or kVA of maximum demand plus so much per unit consumed. These two components are different in kind and must be plotted quite separately. They cannot be added except on the basis of some known load factor, yet they need to be plotted side by side since they are affected similarly (though not equally) by such items as losses in the plant and cables. In the type of diagram here shown these two components are plotted vertically below and above a horizontal "zero" line, so that they progress side by side and can at any point be added if the load factor is known.

A glance at the diagrams on pp. 216 and 222 will show that they run parallel to this horizontal zero line, and below this line the ordinates (measured downwards) represent annual costs per kW of demand and are scaled in pounds sterling. The ordinates above the line represent the cost per kWh and are scaled in pence. (In what follows this is referred to as the "energy" scale.) The left-hand side of the diagram may be said to represent the generating station and the right-hand side the consumers' premises.

In order to illustrate the use of this diagram, two cases are taken, differing as much as possible. The first is a simplified generalisation of the whole of electricity supply in Great Britain in the year 1922–23, for which purpose generation, transmission and distribution are largely grouped together. It endeavours to analyse and explain the actual mean energy price charged in this country during that year, and to show what it might have been had the load factor been different. The second case taken is treated in more detail and is a particular case representing one typical station and employing present-day costs. In this case a hypothetical tariff is built up to represent the cost of manufacture.

Supply in Great Britain (1922–3).—The figures employed are taken from the comprehensive statistical return * of the Electricity Commission and are briefly summarised below. In one or two cases round numbers are given, and there has been slight modification of some of the totals in order to avoid duplication owing to the inter-sales of energy between different authorities.

Referring to the table, the fuel cost and other working expenses for the year are given directly in the Return, and can without difficulty be expressed respectively as a charge per unit and a charge per annum per kW of demand on the stations. With regard to the capital expenditure, this must also be expressed in terms of its annual value, and in the Return it will be found that the excess of the total revenue over the total working expenses amounted to just 10 per cent. of the total capital invested. If, therefore, a uniform percentage of this value be assigned to each of the capital items, and if all running expenses are entered at the values shown, the total annual expenditure thus obtained will just equal the total income, which in turn must equal the number of units sold multiplied by mean price per unit.

It should be realised that this simple method of apportionment, although convenient as a first approximation, by no means represents the annual cost of the different items correctly. As regards the strict interest and depreciation charges, these should vary according to the life of the items, and in any case

* Published in June, 1925. A later return has since been published, but for the purpose of a first example the earlier set of figures has been taken, and this has the advantage of increasing the contrast between the two cases. On p. 286, Fig. 27, the later years are illustrated by means of the same sort of diagram.

should average less than 10 per cent., since with interest reckoned at 5 per cent. and a life of twenty years the total figure should be only 8 per cent. The difference may be regarded as profit, but whereas with company undertakings it is distributed as such, with municipalities it is largely used for debt redemption. Thus it will be found from the Return that the municipal undertakings (comprising two-thirds of the whole), whilst showing a similar excess of receipts over expenditure, only distributed 34 per cent. of this surplus in interest and rate contributions, whereas the company undertakings distributed 55 per cent. of their surplus in interest and dividends.

DATA FOR THE YEAR ENDING DECEMBER 31ST, 1922, OR MAY 15TH, 1923.

Aggregate of maximum loads on all stations of authorised undertakers	$1{\cdot}83 \times 10^6$ kW*
Total units generated	$4{,}500 \times 10^6$ kWh
,, ,, sold (16 per cent. less) . . .	$3{,}780 \times 10^6$ kWh
Mean selling price per unit	2·07*d*.
Fuel cost per unit (2·8 lb. at 20*s*. 5*d*. a ton for coal) .	0·31*d*.

	Annual Cost per *kW* of Demand Load on Stations. £
Other generating costs (0·23*d*. per unit $\times$ 4,500 $\div$ 1·83)	2·32
(Total working expenses were £19·7 $\times 10^6$.)	
Working expenses of distribution (12·1 per cent. of 19·7 $\div$ 1·83)	1·30
Rents, rates, and taxes (11·1 per cent. of 19·7 $\div$ 1·83)	1·19
Management (12·5 per cent. of 19·7 $\div$ 1·83) . . .	1·34
(Total capital expenditure at end of period was £158 $\times 10^6$, which is to be reckoned at 10 per cent. per annum.)	
Land, buildings, etc., for generation (13·2 per cent. of 15·8 $\div$ 1·83)	1·14
Plant and machinery for generation (33·4 per cent. of 15·8 $\div$ 1·83)	2·88
Land, buildings, etc., for transmission (1·8 per cent. of 15·8 $\div$ 1·83)	0·15
Plant and machinery for distribution (11·9 per cent. of 15·8 $\div$ 1·83)	1·03
Transmission lines, mains, services, etc. (39·7 per cent. of 15·8 $\div$ 1·83)	3·43
Total standing charge	£14·78

* The total kW installed was 69 per cent. greater than this.

Hence it will be seen that in apportioning a uniform 10 per cent. on all capital items, this is merely a rough and convenient way of accounting for the money actually made rather than a scientific representation of real cost. In the second example taken, a little more detail will be employed in this connection.

In the diagram, Fig. 14, the various expenditures on account of generation, transmission, and distribution are all grouped together on the left of the diagram ; and using the figures tabled above, a total standing charge is obtained of £14 15*s.* 7*d.* per annum per kW of demand load, and is plotted downwards below the zero line to the scale shown. The fuel cost at the same point is 0·31*d.* per unit generated, and is plotted above the zero line to the " energy " scale.

Of the total units generated, the Return shows that only 3,780 millions were sold, the difference, namely, 16 per cent., presumably being lost in conversion, transmission, and distribution. Hence the price per unit to the consumer is increased in the corresponding ratio and becomes 0·37*d.* (top right-hand side of diagram). This loss, since it means an increase in the cost per unit, can be represented by a cost tributary coming into that portion of the diagram above the zero line, and since it takes place somewhere on the journey between station and consumer, it is shown in an intermediate point between left and right.

Turning now to the portion of the diagram below the zero line, it will be found that the transmission losses decrease the kW capacity to meet consumers' demand, just as they decrease the available units, but not quite in the same ratio. This is due to the fact that with apparatus having iron and friction losses these items cause a loss of energy throughout the twenty-four hours, whereas the kW capacity is affected only by full-load conditions. Thus the capacity to meet demand is reduced only in the ratio of the full-load efficiency whereas the energy is reduced in the ratio of the all-day efficiency, which will be less (except with cables) by an amount depending on the low load losses and the smallness of the load factor.

In the example shown, the reduction in kW capacity is taken to be 13 per cent.,* and the price per kW of demand is correspondingly increased to a figure of £17 0*s.* 7*d.* per annum. This is represented by a tributary coming into the lower portion

* This figure is tentative and is not based on any data in the Return.

of the diagram at an intermediate point between station and consumer.

The aggregate maximum demand on the stations has been given above, and from this and the total units generated, the mean load factor for the country is found to be 28 per cent. The difference between the loss in units and the loss in kW will lower this to 27·2 per cent., and assuming this load factor, the two price components can be added together in the manner described below, and will then be found to total 2·07*d.* per unit, the mean price actually charged throughout the country.

But although these two components may be said to represent the price of the electricity as it is delivered from the stations, they do not correctly represent the price to the individual consumer, because the individual load factor is lower than 27 per cent. owing to the diversity factor. Thus if the sum of all the individual consumers' maximum demands is 25 per cent. more than the maximum demand on the stations, giving a diversity factor of 1·25,* it follows that the average consumers' load factor will be 27·2/1·25 = 21·7 per cent.

The effect of a diversity factor of this amount will be to increase the kW capacity of the station to meet consumers' demands by 25 per cent., and this will lower the annual cost per kW to the consumer by a corresponding amount. This is shown on the diagram by horizontal shading, and it results in a final price to the consumer of £13 12*s.* per annum per kW of demand plus 0·37*d.* per unit of consumption. (Right-hand side of diagram.)

For any particular load factor the annual charge per kW of demand can be expressed as a charge per unit and then added to the working charge to give a total overall price per unit. Thus for the mean consumer's load factor of 21·7 per cent., each £1 per annum demand charge will represent

$$\frac{240}{\cdot 217 \times 365 \times 24} = 0{\cdot}125d. \text{ per unit, and on the diagram the}$$

lower scale has been so chosen that, at this load factor, distances on it represent prices per unit to the "energy" scale. Thus the final demand charge of £13 12*s.* represents, at a load factor of 21·7 per cent., 13·6 × ·125 = 1·7*d.* per unit, which added to 0·37*d.* gives the mean price actually charged, namely, 2·07*d.* Hence for this particular load factor the total distance CN represents graphically the total charge per unit (to the "energy"

* This figure is tentative and is not based on any data in the Return.

scale), the total being composed of an energy charge CO plus a demand charge ON.

The diagram therefore represents a two-part tariff which is capable of looking after the total costs of supply, as deduced from the Commissioners' Return, whatever the load factor of the consumer. It also represents to scale this same cost expressed as a single charge per unit for a consumer having a particular load factor, the charge thus obtained being the actual mean charge per unit for the whole country, and the load factor being the mean load factor for the country as deduced from the Return.*

To find the price per unit which should be charged on the above assumption to a consumer having a load factor of, say, 33 per cent., the annual charge per kW of £13 12*s.* becomes

$$\frac{13{\cdot}6 \times 240}{0{\cdot}33 \times 365 \times 24} = 1{\cdot}13d.,$$ which when added to 0·37*d.* gives

an overall price of 1·5*d.* per unit. Thus if a consumer having this load factor and paying this price per unit wishes to obtain a clear conception of how his price is made up he can obtain it from a diagram such as the above, but in his case the figure and scale below the zero line must be reduced to two-thirds of its present size in order to bear the right proportion to the upper part when the whole is scaled in pence per unit. Similarly, if the load factor is 10 per cent. the overall price should be 4·1*d.* per unit, and for such consumer the lower part of the diagram must be more than double its present size in order to add to the upper portion on the existing "energy" scale.

It is difficult, with the data available, to determine precisely what should be regarded as the cost of generation, but by adding to the total working and capital expenses of generation a proportion of the management and taxes calculated on the ratio of the corresponding capital expenditures, the point T′ is arrived at. Then for energy bought at the generating station, assuming that the demand of such consumers had a diversity factor as compared with the main demand of 1·25, the correct price would be given by RT plus GR, namely £6 2*s.* 6*d.* per annum per kW plus 0·31*d.* per unit. If such a consumer had a load factor of 21·7 per cent., the overall price

* It may be objected that the figures for the diversity factor and for the loss in kW are not obtained from the Return; but whilst this would make a difference to the annual charge per kW, it will not affect the total charge expressed per unit, which will still add up to the correct figure.

per unit would be represented by the distance GT to the "energy" scale and would total 1·07*d*. per unit—just over half the mean selling price.

It must be clearly understood that the above, although representing the total costs of supply for the country on a two-part basis, by no means represents the price to be charged. Even if a universal two-part tariff could be framed it would obviously have to employ a much lower kW charge than the high figure shown above, particularly for the low load factor consumer owing to his higher diversity factor. The diagram represents a simplified allocation of costs and receipts rather than a possible tariff; moreover, it is a generalisation covering a wide range of variations, small stations as well as big, and domestic consumers paying a high overall figure per unit as well as power consumers paying on a two-part basis. Of the three main variations concerned, load factor, diversity factor, and distribution costs, only the first named is differentiated in the diagram, the other two being averaged over the whole country.

As regards the differentiation to cover transmission and distribution expenses, and the framing of possible tariffs at the various points, a single case is taken below and treated in detail. This follows the same lines as the previous one, but generation, main transmission, and distribution are here taken separately, and the diagram as it develops from left to right represents to some extent the spatial arrangement of stations and gear.

Single Modern System.—The data here employed is taken from a paper by S. J. Watson, presented to the Manchester Association of the *Institution of Civil Engineers* on January 27th, 1926. A single station of 100,000 kW capacity is postulated and a complete system of generation, main transmission, and distribution is worked out. Using present-day cost figures a two-part tariff is indicated for each point in the system, capable of covering all the costs up to that point.

The figures are taken exactly as they stand in Mr. Watson's paper, with the object of showing how they or any similar set of figures can be usefully represented in a diagram of this kind. They were selected as being one of the clearest and most detailed analyses of the cost of electricity supply that has recently been made.

As regards generation (which includes transformation up to

33,000 volts), 95 per cent. of the cost of the fuel, oil, etc., rank as running expenses proportional to the energy generated, and are paid for per unit. The remainder of the fuel and all the other generating costs are reckoned as standing or preparatory costs to be paid for by an annual charge per kW of demand.

Main transmission takes place at the above-mentioned pressure to sub-stations where the supply is transformed down to 6,600 volts for secondary transmission. At this point the supply is considered to be split up into several portions, only one of which is here shown,* namely, that to rotary sub-stations for conversion to three-wire, D.C., 440 volts between outers.

MAXIMUM STATION LOAD 70,000 kW STATION LOAD FACTOR 40 PER CENT. (THE TOTAL INSTALLED IS 100,000 kW.)

Annual Cost (÷ £1,000).		Standing Item.		Working Item.	
		kW available (÷ 1,000).	Annual Cost per *kW* of Demand.	Annual Units delivered (÷ 10^6).	Cost per Unit (Pence).
Fuel, etc.—95 per cent. of £134,000...		...	...	245	0·125
5 per cent. of £134,000...	7				
Wages, rates, etc. ...	64				
Station first cost (at 10 per cent. per annum)	150				
	221	70	£3 3 0		
Diversity factor main cables of 1·10		77	2 17 3		
Cost of main cables and sub-stations (at 9 per cent. per annum) ...	40·5				
	261·5				
Losses in main cables and sub-stations (kW reduced $2\frac{1}{2}$ per cent., units $3\frac{1}{2}$ per cent.) ...		75	...	236·4	0·129
Diversity factor 6,600-v. cables of 1·05		79	£3 6 2		

* From here onwards the figures of the original paper are doubled as though the whole distribution were D.C. With regard to the D.C. voltage it should be noted that the approved standard for all future work is 460/230.

Annual Cost (÷ £1,000).	Standing Item.		Working Item.	
	kW available (÷ 1,000).	Annual Cost per kW of Demand.	Annual Units delivered (÷ 10^6).	Cost per Unit (Pence).
Cost of cables, etc. (at 8 per cent. per annum) ... 32				
293·5				
Losses (1 per cent. reduction in kW and units)...	78	£3 15 0	234	0·1307
*Cost of rotary subs. (at 10 per cent. per annum) 80				
Running expenses of ditto 24				
397·5				
Losses (kW reduced 7 per cent., units 11 per cent.	72·6	...	208	0·147
Diversity factor D.C. mains of 1·10 ...	79·8	£4 19 5		
Cost of D.C. mains £20 × 79·8 (at 11 per cent. per annum) ... 175·6				
573·1				
Losses (7½ per cent. reduction in kW. and units)	73·8	...	192·4	0·159
Diversity factor individual consumers of 1·20 ...	88·6	£6 9 2		

The main figures are summarised in the foregoing table. These are necessarily greatly condensed, being inserted merely as a check on the diagram (Fig. 15), and any details and explanations required should be sought in the paper from which they were taken. In the double columns are shown respectively the kW and units available at each point and the corresponding prices per annual kW and per unit.

Commencing at the left-hand side of the diagram, Fig. 15, the following points should be noticed. In the first place the cost of generation and transformation to 33,000 volts is found by adding the total expenses at the power station, these being expressed on an annual basis. For this purpose the whole of the plant and equipment is reckoned at a uniform figure of 10 per cent. per annum on the capital cost, and it will be seen that roughly one-third of the first cost went in land, buildings,

etc., one-third in the steam plant, and one-third in the electrical gear. As a figure of 8 per cent. would cover interest at 5 per cent. and repayment in twenty years, and since at least the first of the three groups of expenditure would not require renewing so soon as this, the allowance of 10 per cent. should provide a reasonable margin for insurance and reserves against any unexpected change in the fortunes of the supply authority or the requirements of the public.

Assuming that the sum of the individual maximum demands on the main cables reaching the central station is 10 per cent. more than the aggregate demand which they combine to make on the station, the generating cost becomes £2 17*s*. 3*d*. per annum per kW of demand plus 0·125*d*. per unit; and this is the cost price which could be charged to a consumer buying at the power station terminals, assuming that his cable showed the above diversity factor to the main body of the demand. In the diagram (Fig. 15) the two scales have been so chosen that, for a load factor of approximately 25 per cent., ordinates on the under side of the diagram represent pence per unit to the "energy" scale; so that for this load factor the dotted bracket represents to scale the generating cost expressed as an overall price per unit, and this equals 0·44*d*. per unit.

From here the diagram works to the right and shows the effect in turn of E.H.T. cables, main sub-stations, 6,600-volt cables, rotary sub-stations, and D.C. distributing mains. For each of these items there is a loss both in energy output and in kW capacity to meet demand, so that the price per unit and the annual price per kW are each correspondingly increased. There is also the capital cost of the various distribution gear, reckoned at percentages from 8 to 11 per annum, and this serves to swell the lower part of the diagram.

It will be noticed that since the cost per unit or per kW is in the form of a quotient—Total Cost/No. of Units or kW—this may be increased either by an increase of the numerator (greater cost) or a decrease of the denominator (*i.e.*, by losses). The former is shown on the diagram by shaded additions to the cross-sectional area of flow, the latter by curved tributaries flowing into the main stream. (A more exact interpretation ld be to regard the total volume of money starting at the and flowing leftwards, being spent or dissipated in the s ways shown.)

ect of a diversity factor is to increase the number of ble to meet demand, owing to that demand not all

taking place simultaneously. This increases the denominator of the above quotient, and so reduces the cost per kW. On the diagram this is indicated by horizontal shading, and it takes place wherever a number of cables meet at one point, *i.e.*, at each station or sub-station, and also when a number of individual consumers' lines enter the street mains and distribution pillars.

The composite price at each stage of the process is shown on the diagram to the two scales marked, and the final price to the D.C. consumer is £6 9*s.* 2*d.* per annum per kW plus 0·159*d.* per unit. In the original paper the power station is taken as working at a load factor of 40 per cent., but owing to the various diversity factors and to the fact that the constant losses reduce the units more than the kW capacity, the average load factor of the individual consumer is reduced to 24·8 per cent. The two scales on the diagram have therefore been chosen so as to correspond at a load factor of approximately 25 per cent., and for a consumer having this load factor the overall price per unit is represented by the total distance CON to the "energy" scale and equals 0·87*d.* per unit. For any other load factor the lower portion of the diagram must be multiplied by the ratio 0·25/new L.F. before adding it graphically to the upper portion, in order to find the total overall price per unit.

Conclusions from Diagram.—Some of the points which the diagram particularly illustrates may be briefly mentioned. The cardinal feature is, of course, the high proportion of the standing charges, and the influence thereon of the load factor. Even when scaled, as in the above cases, for a fairly high individual load factor, the enormous preponderance of the lower portion of the diagram is very striking. Taking the price to the low tension consumer, it will be found that in each of the above cases the demand charge when expressed per unit is four and a half times the energy charge at the average consumer's load factor (22 to 25 per cent.). Hence for a purely lighting consumer, whose load factor may easily be only half this, the demand component will form 90 per cent. of the whole price: and for any lower power factors the energy item would be almost negligible, and a charge based purely on maximum demand would be justified.

Comparing the selling with the generating cost, in each of the cases taken above the demand component is rather more than

double, the energy component very much less than double. The net effect (with the "average" load factor) is to give an overall price to the low tension consumer almost exactly double that of the price at the power station. But the total is differently made up and will therefore be somewhat less than double on a higher load factor, and more than double on a lower load factor, than 22 to 25 per cent.

Another point to notice, in the second diagram particularly, is the way in which the transmission and the distribution costs mount up in a steadily increasing ratio as the supply proceeds. The earlier, higher voltage stages not only show much smaller energy losses (as would be expected), but even as regards capital charges they appear far less serious than the later items, and it is abundantly evident that low tension distribution, and particularly of direct current supply, is a most expensive luxury. As regards the possibility of cheapening by the employment of A.C. throughout, the main saving would be in the low tension sub-stations. In the paper from which these figures were obtained an alternative is worked out for static stations delivering to consumers at 400 v. and 230 v. A.C. This is lightly sketched on the right-hand portion of the diagram (with dotted lines), and it results in a final price of £5 13*s.* 11*d.* per annum per kW of demand plus 0·150*d.* per unit (C′ON′).

A further use for the diagram is in showing the effect of changes in the price of the different "ingredients." The price changes which are likely to produce the biggest effects are changes in the cost of fuel, in the cost of electrical plant, and in the cost of capital. Changes in the smaller items, such as materials other than fuel, wages, salaries and management, rates and taxes, will have less effect on the total.

As regards fuel, a change in its price will produce an exactly proportional change in the whole of the upper portion of the diagram—thus comparing the two examples above, the generation for the country in 1922, if transferred to the present day with fuel at 16*s.* instead of 20*s.* 5*d.*, would result in a price per unit of 0·29*d.* (In comparing this with the figure of 0·159*d.* it must be remembered that generation for the whole country includes many small and inefficient stations, and it also includes profits in some cases.) Changes in the cost of electrical plant and machinery, if accompanied by similar alterations in the smaller items, could be shown by a proportional change in the whole of the lower part of the diagram.

The cost of capital, as evidenced by the rate of interest which has to be paid, is in a somewhat different category from the other items. Thus if the prices of all the items were to alter in a certain ratio, and the diagram were altered in this ratio, a change in the rate of interest would make a further difference over and above all the other changes, but it would affect the lower part of the diagram only. The change would not be a proportional one, even on the purely capital charges, since interest and depreciation would be affected in opposite directions. The result of a change in the rate of interest could therefore only be found by recalculation throughout.

It will be noted in conclusion that no mention has been made of power factor, and in fact a D.C. distribution was selected in the second diagram partly with a view to avoiding this item. The economic effects of a bad power factor are in many respects similar to those of a bad load factor, and are dealt with in detail in the chapter which follows.

Typical Power Tariff.—Although the division of costs into two components can usefully be applied to any part of a supply system, and is much the simplest way of representing the economics of the case with some degree of accuracy, it does not follow that the tariff will necessarily take this form, since many other factors enter besides pure economics. To the domestic consumer, who dislikes technicalities even more than the power user, and whose consumption is not sufficient to justify elaborate metering, it is more usual to supply electricity at a flat rate (or two different rates for different purposes); the price being then high enough to cover all the possible expenses involved, even assuming a purely lighting load factor. To the large-scale power user, who is more likely to understand the meaning and importance of load factor and who is more likely to be able to effect an improvement, a two-part tariff is very frequently put forward, although here also a higher flat rate is generally offered as an alternative.

The following is a typical two-part power tariff: £5 per annum per kW of maximum demand plus $\frac{1}{2}d$. per unit consumed $\pm$ 0·03d. per unit for each 1s. per ton rise or fall in the price of a specified grade of coal, now standing at 14s. a ton. The maximum demand to be measured by a maximum reading ammeter or other device taking not less than twenty minutes to come to its full reading and over periods not exceeding three months. With such a tariff the lowest overall price per unit

will naturally be paid by a consumer whose load factor is 100 per cent., and will be $\frac{5 \times 240}{365 \times 24} + \frac{1}{2} = 0{\cdot}137 + 0{\cdot}5 = 0{\cdot}637d.$ per unit. For a consumer whose load factor is F (F being fractional) the total cost per unit will be $\frac{0{\cdot}137}{F} + \frac{1}{2}d.$, so that it will become $1{\cdot}87d.$ per unit when the load factor is 10 per cent.,

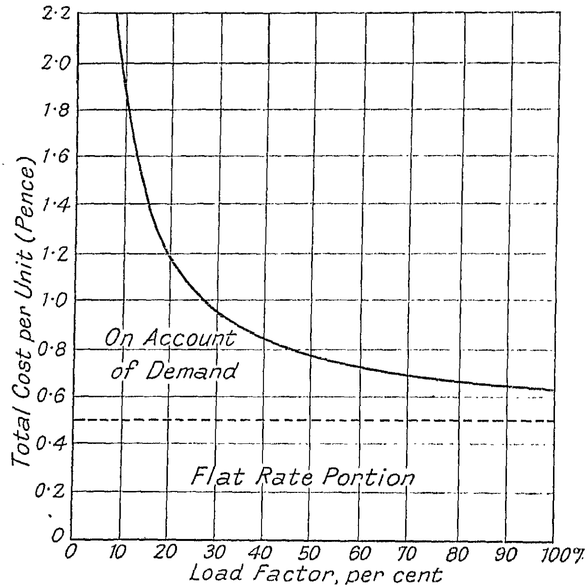


FIG. 16.—Cost per Unit with Two-Part Tariff.

and so on. In Fig. 16 the total cost per unit with this tariff is shown for a range of load factors from 8 to 100 per cent. The curve is made up of two parts, separated in the figure by a dotted line, namely, the flat rate portion of $0{\cdot}5d.$ plus a rectangular hyperbola on account of demand, the latter rising as the load factor falls.

Cost per Unit on Two-Part Tariff.—When the price at various points in a system is known in the form of a two-part tariff, it is useful to be able to show this as a single overall figure per unit for consumers of various load factors. Taking a complete supply system such as that illustrated in Fig. 15, the price at

any particular point can be represented for all load factors by the sum of a straight line and a hyperbola as shown in the last figure. On a large system this would mean an elaborate set of curves, difficult both to draw and to read. For such purposes it is therefore convenient to use graph paper in

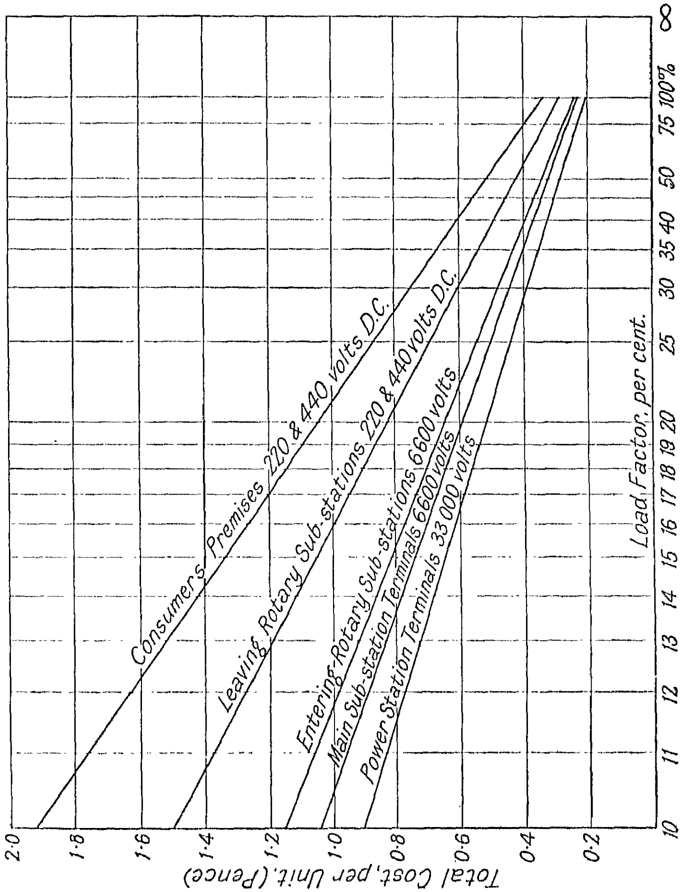


FIG 17.—Cost per Unit at Various Points.

which the base scale is an inverse or reciprocal one, *i.e.*, on which the distances from the right-hand side are the reciprocals of the figures scaled along the base. Thus in any costing problem, if the price per article or per unit is made up of two components, one constant (*e.g.*, working cost) and one inversely proportional to the number made (standing cost), the total

price per unit when plotted on ordinary graph paper will consist of a horizontal straight line plus a rectangular hyperbola, but if plotted on paper with a reciprocal base scale it will consist of a horizontal straight line plus a falling straight line as in the figure described below.

Graph paper ruled in this way has recently been put on the market under the title "costing paper,"* and in Fig. 17 this is used to illustrate the hypothetical supply system whose data is given in the table on p. 218 and in Fig. 15. This table shows the price on a two-part basis at five different points on the system, ranging from the power station terminals at 33,000 volts A.C. down to the consumers' premises at 220 volts D.C. The five straight lines on Fig. 17 represent these five points. At any point the two-part tariff can be resolved into a single overall price per unit provided the load factor is known, and the graph shows this for all load factors (base) from 10 to 100 per cent. It will be seen that these curves give similar information to that given by the previous figure, but owing to the reciprocally scaled base they are straight lines instead of hyperbolæ. [Note on Construction.—In order to draw one of these lines it is, of course, only necessary to know any two points on it. Referring to the table on p. 218, the right-hand column gives the working cost per unit (which constitutes the energy portion of the tariff) at the five points, and this is scaled on the line marked "infinity" on the extreme right of the graph. The total cost per unit at any one load factor, *e.g.*, 10 per cent., must then be found, this being made up of the energy cost plus the standing charge (£) multiplied by $\frac{240}{0{\cdot}1 \times 365 \times 24}$. This can then be scaled on the line corresponding to 10 per cent., and the two points when joined will give the line required.]

* Published by A. West and Partners, 36, Broadway, Westminster, S.W. 1.

CHAPTER XIII

POWER FACTOR

Definition.—When the pressure applied to and the current in a circuit are both steady, the power in it is found from the product of the pressure and current, but when pressure and current are varying, the power cannot be calculated in this simple manner. Thus, when the pressure and current are alternating they can each be represented by their virtual (root mean square) values, but the power will usually not be found from the product of the two R.M.S. values, but will be less than this product. The power in the direct current circuit could be likened to a perfectly trained boat crew or a perfectly made cord—all the rowers or all the strands are "pulling together," so that the efficiency of the arrangement is 100 per cent., *i.e.*, the composite result is the maximum that can possibly be extracted from all the elements making it up. In an alternating current circuit, the current either does not follow the exact shape of alternation which the pressure maps out, or it does so at a slightly different time, or both; so that the two are not pulling together perfectly, and occasionally one is actually pulling against the other.

The power at any instant is, of course, the product of the pressure and current at that instant, and if all these instantaneous powers are summated, their average value over any period gives the true or effective power (watts or kilowatts, kW). When the effective R.M.S. values of pressure and current are multiplied together this product is called the apparent power (volt-amperes or kilovolt-amperes, kVA). The ratio true power $\div$ apparent power is called the power factor.

The pressure on most modern supply systems follows very closely a sine wave in its alternations, and the current follows the same shape more or less closely according to the character of the circuit, but usually lagging behind the pressure. When pressure and current both follow sine waves they can be represented by the projections, upon a straight line, of vectors of constant length, rotating at the same constant speed but with a definite angle between them. This is called the angle of lag,

when the current is behind, or lead when it is in fron usually denoted by the symbol ϕ. In such cases th factor is equal to the cosine of the angle of lag or l true power (kW) = apparent power (kVA) $\times \cos \phi =$ $I_{R.M.S.} \times \cos \phi$.

When the wave shapes are irregular, or when th polyphase circuit in which the different phases are not l the power factor is not always easy to find or even define. But it is believed that the above definition c case as well as a simple one can, and will be sufficien present purpose. Furthermore, it will be assumed follows that the pressure and current both follow sino

Causes of Bad Power Factor.—The chief causes o scale of bad power factors are the following :—

Induction Motors.—Nearly 70 per cent. of th supplied by public authorities in this country is f purposes, and the polyphase induction motor is b simplest electrical machine developing mechanical p is relatively cheap, has a good efficiency, and req minimum of upkeep, and there is no immediate like its becoming any less widely used than it is at pres full-load power factor is usually of the order 80 to 90 and on lower loads considerably worse.

Transformers.—When on full load the power factor efficiency, is very little short of unity, but since the ma current is practically constant at all loads it represen bigger proportion of the total when the transformer loaded, so that the power factor is then a poor one. tension distribution system it is often necessary to l numbers of unattended transformers permanently to the line, and during light load periods this results low power factor.

Electric Furnaces.—When the station serves electro and metallurgical industries the power factor is ofte two reasons. In an arc furnace the arc itself ha conductivity at the commencement of the cycle, so current is delayed with respect to the voltage wav addition it is often necessary to employ reactance minimise the dangers of short circuits. A plain furnace may have quite a good power factor, but i of the induction type it is usually bad, owing to irc forming the secondary.

Current Limiting Reactances.—Apart from their use in connection with furnaces, choking coils are frequently employed in central stations to minimise the current in the event of faults, and these are a further cause of bad power factors.

Transmission Lines.—Although on light loads these act as condensers and take a leading current, this may be more than neutralised on load by the inductive drop caused by the load current. The effect is small on multicore cables, but relatively large with an air line.

It will be noted that in all the above cases the power factor is bad owing to the current lagging behind the pressure. A leading current is just as bad, but far less likely. It may arise owing to long sections of electrified but unloaded cable, or where static condensers are left connected during light load periods; but in the great majority of cases the utmost that such leading currents can do is partly to neutralise the much more extensive lagging apparatus. In what follows it will in all cases be assumed that the correction required is that due to a *lagging* current.

Methods of Improvement.—Power factor improvement can be carried out in any one of the three following ways :—

(1) By the installation of apparatus taking a leading current, the installation being made solely for the purpose of power factor improvement.

(2) By the use of apparatus capable of taking a leading current, but which also performs other useful functions.

(3) By the improvement of apparatus employed for other purposes in such a way as to decrease or eliminate its lag.

Methods (1) and (2) are capable of effecting very great improvement in the power factor of a lagging system, since they apply positive correction by injecting a leading current capable of neutralising the lagging currents due to other apparatus. Method (3) can in general only reduce the amount of the lag; it cannot neutralise it or cause a lead. In order to be effective it must be very generally practised, since an improved machine can only solve its own problem and does little or nothing to assist other lagging plant.

The simplest example of the first method is the static condenser, consisting of Mansbridge plates, comprising a length of paper coated with tin so as to have a considerable electrostatic capacity. Usually each unit has a capacity of approximately one microfarad, the units being assembled in frames fixed in

boiler plate tanks and completely oil-filled. The kVA capacity is proportional to the frequency and to the square of the pressure, but with the materials and thicknesses generally employed the most suitable working pressure is about 600 volts, so that on circuits of moderate pressure it is advisable to step the pressure up or down to this figure by means of transformers. For higher pressure circuits (*e.g.*, up to 3,300 volts) condensers are made for direct connection, but this involves a discharge resistance and more expensive switchgear.

The losses on such a condenser are extremely small (about $\frac{1}{2}$ kW per 100 kVA output unless transformers are employed), so that its current leads the voltage by almost 90 degrees. As there are no working parts, its maintenance and depreciation are extremely slight, and the equipments that are in use have given little or no trouble. The first cost of equipments for a fifty-period 400 to 440-volt circuit and ranging in size from 100 to 300 kVA works out at about £3 per idle kVA.

The best example in Class (2) is an over-excited synchronous motor. When compared with the induction motor as a driving unit it is more expensive, and has a smaller "pull-out" torque, but by over-exciting its field system its current can be made to lead to such an extent that it will counterbalance the lag due to several other machines. Its performance in this respect is so good that it will frequently pay to install it for this sole purpose, and to run it light without utilising its mechanical output. It then comes into Class (1), and is a competitor of the static condenser, being then termed a synchronous or rotary condenser. A comparison between these two when employed for this purpose will be found on pp. 266—270.

When used as sources of mechanical power, in which cases ease of starting is of importance, these motors are usually fitted with windings so that they start up as induction motors, and then pull into synchronism when the excitor is switched in. Such machines are termed auto-synchronous motors or synchronous induction motors.

Another machine in this group is the over-excited rotary converter, which is somewhat similar in this respect to the synchronous motor. Unfortunately, owing to the disposition of the currents in the armature of a converter, the effect of over-excitation is greatly to increase the losses. Hence this method is uneconomical as a means of injecting leading kVA, and stands no comparison with the others outlined above. At

the same time, rotary converters may be very useful as a form of unity power factor load, thus helping to bring up the general average; and a supply system having a number of rotary sub-stations can look for some appreciable degree of power factor improvement on this account.

The best example of Class (3) is an induction motor fitted with a phase advancer. This involves the use of a slip-ring machine in place of the more usual squirrel-cage type, and the slip rings are connected to either a rotary (Leblanc, Scherbius, or Walker type) or an oscillatory (Kapp type) phase advancer. The effect of this appliance is to introduce an E.M.F. of slip frequency in the correct sequence into the slip rings in such a way as to shift the rotor current forward. Owing to the inter-action between stator and rotor, this lead is translated into the stator circuit, but on a magnified scale. Neglecting losses, which are quite small, a 6 or 7 kVA phase advancer working on a large motor will benefit the line to the same extent as a static or rotary condenser giving out 250 kVA. At the same time, the overload capacity of the motor is increased and there is no loss in efficiency.

Another way of looking at it is to say that by means of the phase advancer the magnetising current of the machine (which in an induction motor, just as in a lightly loaded transformer, is the cause of bad power factor) is supplied through the rotor instead of the stator. Since magnetising kVA is proportional to frequency, and since the frequency in the rotor is only 2 or 3 per cent. of the supply frequency, it follows that motors excited at this lower frequency require only a fraction of the kVA necessary when they are magnetised in the usual manner. Moreover, by *over*-magnetising them, the stator current can actually be made to lead slightly.

It is possible to fit a phase advancer on any slip-ring induction motor, although the best results are obtained by designing the motor with this in view. The cost of such a motor, including the phase advancer, is still less than that of a synchronous motor, whilst the efficiency, overload capacity, and other running characteristics are better. On large driving units this is undoubtedly the cheapest method of improving the power factor, but where a number of small induction motors are employed it is preferable to neutralise their lagging currents as a whole by means of a separate large unit.

As regards other examples of this class, most of the machines enumerated as causes of bad power factors can be improved to

some extent by careful design. Thus in an induction motor the air gap can be reduced by fitting ball or roller bearings, and this materially improves the power factor. A similar improvement can sometimes be made in transformers by the employment of interleaved instead of butt-jointed cores; and in a system employing a number of small transformers of this type the power factor when lightly loaded can be considerably improved in this way.

A recent development which can be considered as coming into the third of these groups is the compensated induction motor, of which the No-lag machine of Messrs. British Thomson-Houston is an example. This motor has a small commutator by means of which the phase position can be controlled, and as usually arranged the power factor is about 0·9 leading at no load and unity at full load. Such a machine is cheaper than an auto-synchronous motor, and can be built for small sizes, but it is naturally dearer than a plain induction motor and usually somewhat less efficient. An example illustrating the economics of such an installation is given at the end of the chapter.

A final point which must be considered in comparing different methods of power factor improvement is the question of where the improvement plant should be situated. Naturally the prevention of lagging currents is better than their cure, and wherever possible the motors or other sources of potential trouble in this respect should themselves be fitted with the necessary correction plant. But where the consumer does not generate his own energy, and where no encouragement is given him in the tariff levied he is hardly likely to go to the necessary trouble and expense of providing gear which is not essential to his particular needs. The supply authority will then have to effect whatever improvement may be necessary or desirable, and the same thing applies where there are a number of small consumers who cannot economically improve their own individual power factors. The plant should then be placed as near as possible to the centre of gravity of the lagging load, *e.g.*, at the sub-station, thus reducing the losses and voltage drops in the transmission lines.

Magnitude and Effect.—Unless phase-advancing plant is installed, the power factor of a station supplying a mixed load will usually be less than 80 per cent., and may be as low as 65 per cent. Even at the higher of these figures this means

that a very appreciable portion of the capacity of the alternators, cables, etc., is wasted in carrying wattless current—current which performs no useful work, but heats up the conductors. Thus if the power factor could be improved from 80 per cent. to unity, the capacity of the whole of the electrical equipment could be raised by 25 per cent. without the addition of a single machine, switch or cable.

The effect of a bad power factor is very similar to the effect of a bad load factor considered in the previous chapter. In each case the consumer is demanding apparatus which he is not utilising to the full energy output of which it is capable. So that if metered and charged only on energy consumption, he will be putting the station to an unrecompensed capital expense which will be greater the poorer his load factor. There are, however, the following differences. With power factor there is comparatively little in the nature of diversity factor, since a leading power factor on a large scale is practically unknown, and such diversity of phase angle as does exist (see p. 236) is not very considerable. A bad power factor demands bigger cables and electrical plant for a given energy consumption just like a bad load factor, but it does not demand bigger steam or prime mover plant. On the other hand a bad power factor causes heating losses, so that in addition to the standing charge it affects to some extent the running charge also. It also has a serious effect upon the voltage regulation.

Cost of Low Power Factor.*—In order to obtain a clear idea of the supply cost resulting from a bad power factor it will be well to start with a purely theoretical situation, namely, a station and supply system in which every item can be varied at will. If this is considered first as supplying a load at unity power factor, and then as supplying the same power, but at a lagging angle, it will be possible to make some estimate of the extra cost involved. It may be supposed that the tariff in the first case is a two-part one of a fixed annual charge per kW of demand plus a uniform price per unit consumed. In what follows, the numerical difference between the apparent and the true power will be referred to as the "extra kVA," and the numerical difference between the apparent and the true consumption as the "extra kVA hours." The object of the present section is then to discover what each extra kVA of

* This analysis, together with the economic calculations which make up the next chapter, is extracted from articles by the author in *World Power*, November and December, 1927.

demand costs as compared with the true kW of demand, and what each extra kVA hour of consumption costs as compared with the cost per unit of the true energy.

The cost of giving a supply to a consumer may be divided up into four groups: standing cost of generation, standing cost of transmission and distribution, working cost of generation, and working cost of transmission and distribution. The first two, if they are to be covered by the fixed charge in the tariff, should include every expense involved in being ready to supply, *i.e.*, practically everything except fuel. The third and fourth items, covering expenses proportional to actual consumption, will consist of 80 to 90 per cent. of the fuel and a few smaller items, such as water, some labour, etc. Dividing on a basis of fuel only, and making reasonable assumptions as to diversity factors, the average cost on a two-part basis of the supply by authorised undertakings in Great Britain in the last year for which full particulars are available (1924) were as follows:—

At the generating station, £5 13*s*. per annum per kW plus 0·28*d*. per unit.

To the consumer, £11 12*s*. per kW plus 0·33*d*. per unit.

These figures, of course, cover small as well as big stations, and since the load is *not* at unity power factor the standing charge would be considerably less if measured per kVA. Moreover, as explained on p. 217, these figures are an average covering both domestic and power consumers, and so are much higher than the average two-part tariff. They are quoted here merely as a guide to the *proportions* of the costs in question.

Standing Costs.—As regards the first two groups, before considering the question of the rate at which extra kVA are to be charged, it is necessary to decide on what basis they are to be measured, since the instantaneous kVA will change with every change in the magnitude or character of the load. But if the consumer's power factor is approximately constant, its effect upon the standing cost of supply will be chiefly felt at the instant he is taking his maximum power. Hence the effect of a bad power factor as distinct from a bad load factor can be assessed by examining his maximum demand over the period in question and seeing how much greater this is than it would have been had he demanded the same power at unity power factor.

In other words, the effect of a bad power factor on the standing charge can be taken as some function of the maximum

apparent power demand as distinct from the true power component of this same demand. This neglects the effect of various items, such as diversity factor and variations in power factor at different times, but it is, nevertheless, a convenient and usual assumption that the damage done by a bad factor, in so far as this affects standing charges, can be assessed on the basis of the maximum kVA demanded.

Taking the first group of costs only, this will include capital expenses, running expenses, rent and rates, management, etc. If the cost of generation in the example depicted in Fig. 15 is examined, it will be found that the capital charges are in three roughly equal parts, of which two, land and steam plant, may be considered as proportional to the true power, and the third, electrical plant, will be proportional to apparent power (kVA). The running expenses can be similarly split up, but a much greater proportion will probably be found dependent upon apparent power, and as regards some items of attendance and management an apparent kVA will cost more than a true kW. As a rough approximation it may be supposed that about half the total expenses in Group (1) will vary with the kVA of demand, the other half being proportional only to true power.

Taking the second group, namely, the standing charges of transmission and distribution, since these are concerned with the electrical end of the system they may be taken as practically all proportional to apparent power; and, again, some of them may be even more than directly proportional owing to the necessity of maintaining the voltage regulation. It will be seen from the figures quoted above that the transmission and distribution costs in this country make a total of just about the same magnitude as the generating costs, so that combining Groups (1) and (2) it may be tentatively suggested that the standing cost per extra kVA is of the order of three-quarters that of a true kW. Thus if there are two consumers each of whose maximum demand is 100 kW, one at unity and the other at 71·4 per cent. power factor (140 kVA), their standing charges should be in the ratio 100 : 130.*

* It will be noted that instead of levying the extra fixed charge for bad power factor on the difference between the kW and the kVA (here called the extra kVA) it can be levied on the full kVA with a corresponding reduction in the charge per kW. Thus an annual charge of £8 per kW plus £6 per extra kVA can be (and would be) more simply expressed as a charge of £2 per kW plus £6 per total kVA. The other method is here used in preference to this because it is desired to show exactly what addition would be necessary to a simple two-part tariff in order to cover the extra cost resulting from a bad power factor.

Phase Diversity of Demand.—There is another point which must be considered. In the above argument it has been implied that a given addition to a consumer's maximum demand increases the station maximum demand by the same amount, but this is only true if the consumer's maximum coincides with the station maximum not only in time, but also in phase angle. It is, of course, obvious that even in a D.C. system an addition to the maximum demands of consumers will produce a smaller increase in the maximum demand on the station, in the ratio of the diversity factor, so that the charge levied per kW of consumer's maximum can be less (except for the effect of losses) than the cost per kW of station plant in this ratio. Diversity factor may thus be regarded as the aggregate effect of this non-coincidence in time of the consumer's maxima, and its existence is fully realised and easily allowed for. What is not generally recognised is that in an A.C. system there is a diversity in phase time as well as in time of day, so that when a charge is made on the consumer's maximum kVA instead of kW, such a charge should be less on this ground than the corresponding cost per kVA at the station. This item is called by the author "diverse phase factor," to distinguish it from the term "diversity factor" in its generally accepted meaning, and it appears to have received very little attention from writers on the subject.

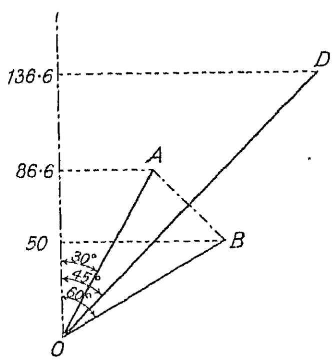


FIG. 18.—Phase Diversity.

Referring to Fig. 18, if the maximum demand on the station (direction OD) lags 45 degrees behind the voltage (*i.e.*, power factor approximately 70 per cent.), and if there are two consumers OA and OB, each with a maximum demand of 100 kVA, but lagging 30 degrees and 60 degrees respectively, then even if their demands coincide as regards time of day with the station demand they would not swell the latter by 200 kVA, but only by 193 kVA. The diverse phase factor in such a case would be 200/193 = 1·035.

Diverse phase factor can be measured in the same way as diversity factor, but by imagining all demands to coincide in

time. The sum of all consumers' simultaneous kVA demands divided by the kVA demand coming on the station would then measure the diverse phase factor ; or it could be defined as the numerical sum of these simultaneous demands divided by their vectorial sum. On a kVA tariff it is the former which is measured and charged for, and the latter which represents extra costs, so that the consumers are to this extent overcharged. The difference is not great in proportion to the whole power, but as it only occurs when kVA are charged for instead of kW, its incidence is entirely on the "extra kVA," and in proportion to this it is quite an appreciable item. (It is for this reason that the factor should be considered separately, and not merely grouped in with the ordinary diversity factor.) In the case cited above, the extra kVA's of the two consumers are 13·4 and 50, making a total of 63·4, whereas the extra kVA coming on the station is only 56·6.

In such a case, and with the figure suggested above for the cost of extra kVA on the system, namely, 75 per cent. of the cost of a true kW, the correct price to charge a consumer would be 67 per cent. Moreover, the factor becomes greater the bigger the phase divergence between the consumers. Thus had the load represented by OA lagged only 10 degrees, that of OB still being 60 degrees and the joint demand then lagging 35 degrees, it would be found that if extra kVA on the system cost 75 per cent. of the cost of a true kW, the correct figure to charge consumers on their metered extra kVA would be only 57 per cent.

The final conclusion as to the costs of the standing charges is that extra kVA should not be charged at more than three-quarters of the price of true kW, and in some cases not more than about half the price, specially when transmission and distribution expenses are small in proportion, and when there is likely to be an appreciable phase diversity.

Working Costs.—Taking the third group of costs, namely, working expenses of generation, these will be almost independent of power factor; whilst as regards the fourth group, covering the working cost (*i.e.*, losses) of transmission and distribution, these will be proportional to the kVA hours. Hence the question of the cost of the extra kVA hours will turn on the proportions which Groups (3) and (4) bore to each other originally. In the case of public supply in 1924, the losses in transmission and distribution amounted to 15·4 per cent. of

the units generated. If only these losses were affected by power factor the price of one extra kVA hour should be just under one-fifth that of a true kW hour.* But it must be remembered that there are also the alternator losses which are proportional to the kVA hours, and, moreover, owing to bad power factors, sets have frequently to be kept running which could otherwise be closed down. When to these factors are added various smaller ones, such as the cost of the extra excitation, etc., it is probable that a figure of one-fourth is more representative.†

Summary and Modifications.—To sum up the above on a two-component basis it will be seen that if the costs of a bad power factor are to be strictly provided for there must be the following four items :—

On account of True Power	*Extra due to bad Power Factor*
(1) Fixed annual charge per kW of demand.	(2) Fixed annual charge (say $\frac{3}{4}$ as much) per extra kVA of demand.
(3) Running charge per kWh of consumption.	(4) Running charge (say $\frac{1}{4}$ as much) per extra kVAh of consumption.

Turning now to actual situations, there are several reasons why the above theoretical considerations require modifying. As regards the standing charge, when the generating station is actually in existence and designed for some particular power factor, its component parts cannot easily be varied. A common practice is to design a station on the assumption of a power factor of about 80 per cent., *i.e.*, the electrical end will be made capable of carrying 25 per cent. more (apparent) power than the steam end. When the actual power factor is below the designed figure, any improvement will virtually increase

* If the station supplies one extra kVA hour without i[illegible] true energy or its fuel cost, this will increase the kVA hours t[illegible] by $1 - 0{\cdot}154 = 0{\cdot}846$, and will decrease the kW hours to the consumer by 0·154. Hence on this basis true kVA hours should cost $0{\cdot}154/0{\cdot}846 = 0{\cdot}182$ times as much as one kW hour.

† In a paper by E. V. Clark (*Journal I.E.E.*, Vol. 64, p. 627), it is suggested that on a system working at 70 per cent. power factor, all kVA hours might be charged at one-ninth the price of true kW hours. This would correspond to a price for extra kVA hours of just under one-sixth that of true kW hours. On the other hand, in some of the Continental systems employing composite reading meters (such as that of Professor Arno, of Milan) the tariff is true kW hours plus *one-third* of extra kVA hours.

the capacity of the *whole* station in the ratio in which the kVA are reduced, hence the cost of a kVA on this point is not three-quarters, but fully as much as the cost of a kW. For power factor improvements beyond this point there will be much less economic justification, and the "cost" of a kVA (*i.e.*, the amount saved by its extinction) will be much less.

When the authorities putting in the improving plant are not the electrical undertakers, the case will only resemble the one considered above in so far as actual costs are reflected in the tariffs charged. The next step is therefore to review the tariffs employed in this country, in so far as they take account of power factor, and to see how they affect the economic situation.

Actual Tariffs.—A considerable number of supply authorities still base their charges only upon true power and energy consumption, making no extra charge for bad power factors. This may be because their load already approaches the power factor for which the station was designed or because they prefer to rely only upon verbal encouragement and advice to their clients. The weakness of such a practice is that the supply authority has no hold whatever over the type of consumption taken from its mains, although one type will put it to a much greater expense than another.

Where tariffs taking account of power factor are employed in this country these can generally be divided into two groups.*

(A) A two-part tariff consisting of a standing charge either per kVA or per kW (the figure then varying with power factor) plus a working charge per unit consumed.

(B) A true power and energy tariff, which may be either of the two-part or single-part type, with in each case a bonus/penalty if the power factor is above or below some datum figure such as 75 or 80 per cent.

A third type of power factor tariff, rather different from either of these, is common on the Continent. Usually it is of the single rather than the two-part type, and the charge varies with power factor, but not to the full extent of the total apparent consumption. Not infrequently it is in the form of a flat rate per complex unit, the latter being perhaps two-thirds of the true kW hours plus one-third of the apparent kVA hours

* See particularly the summary by E. W. Dorey, *Journal I.E.E.*, Vol. 64, p. 637.

(*i.e.*, extra kVA hours at one-third the price of true kW hours). Special integrating meters have been developed for registering such complex units directly on the dial, so that consumers can at any time see the readings on which their bill is reckoned.

kVA Tariffs.—As regards tariffs in Class A outlined above, the energy is measured by an ordinary watt-hour or cosine meter, and the maximum demand is generally measured by a maximum reading ammeter, which when multiplied by the line voltage ($\div$1,000) gives the maximum kVA demand during the metering period. This is usually all charged for at one definite figure (extra kVA ranking the same as kW), which may be in the neighbourhood of £5 per annum per kVA. When such a tariff operates it is easy to show that a good return can be obtained on capital invested in phase improvement plant in bringing the power factor to a point little short of unity. A good example of a straightforward tariff of this description is that of the Warrington Electricity Supply Department illustrated in the chart, Fig. 19, which is reproduced through the courtesy of Mr. F. V. L. Mathias, M.I.E.E. The purpose of the chart is to enable the overall price per unit to be obtained directly for any load factor and power factor.

The chief criticism which can be levied on such a tariff from the point of view of representing actual costs is that in it the extra kVA are usually charged for at the full rate, whereas they do not in fact cost as much as true kW, but only in the neighbourhood of one-half to three-quarters as much ; and, on the other hand, the extra kVA hours are not charged for at all, whereas they do cost something, which has been estimated above as about a quarter that of true kW hours. There are several justifications for this procedure, the chief of which is the convenience of a simple tariff and the comparative ease of measurement. It will be noted, moreover, that the two errors are in opposite directions, thus tending to cancel, though the extent to which they succeed in doing this will, of course, depend on the proportions of the two charges and the load factor of the consumer.

Taking the mean prices for the country quoted above and at the mean consumer's load factor in this year (22 per cent.),* one extra kVA of demand will mean $0{\cdot}22 \times 365 \times 24 =$ 1,928 kW hours of consumption. Hence the extra resulting

* This allows for both losses and diversity factor. The mean load factor on the stations was 30 per cent.

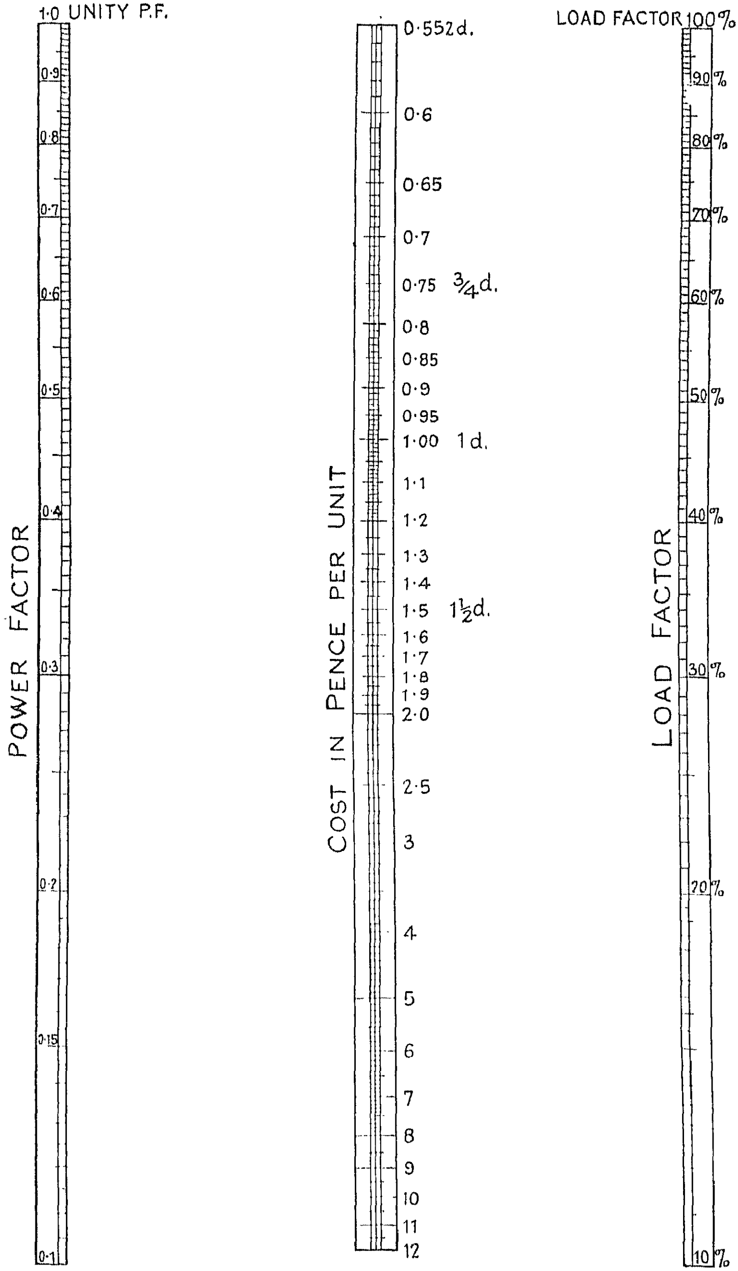


FIG. 19.—Cost per Unit Chart.

Electricity at £4 10*s.* per kVA of Maximum Demand plus 0·43*d.* per unit (coal at 14*s.* per ton).

To find cost per unit place a straight-edge to required Power Factor on left-hand scale, and to required Load Factor on right. Where the straight-edge intersects the middle scale read "Pence per Unit."

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from charging kVA at the full figure of £11 12*s*. per annum instead of at three-quarters of this will be £2 18*s*. per annum, whilst the reduction in not charging for the kVA hours instead of charging for it at a quarter of 0·33*d*. will be $0{\cdot}33/4 \times 1928/240 =$ £0·66 per annum, *i.e.*, the second error will only go a quarter of the way towards balancing the first. There is, however, another reason which makes the above tariff less unfair to the low power factor consumer than at first appears. Usually the demand is measured by a maximum reading ammeter, and this means that the charge is virtually based on the peak load power factor, which is usually better than the mean power factor, owing to apparatus being then more fully loaded.

Another criticism of the Class A tariff is that it gives unequal rewards to different consumers for providing equal services to the supply authority. This can be seen by comparing two adjacent consumers, each of whose maximum demands has a true power value of 100 kW, but whose power factors are 70 per cent. and 89 per cent. respectively. If the former installs phase improvement plant injecting 50 kVA leading by 90 degrees, he will bring his power factor to 88·7 per cent. and reduce his kVA of demand from 143 to 113, a saving of 30 kVA, but if the second consumer installed the same plant, thus bringing his power factor almost to unity, his saving would only be 11·6 kVA. Yet the gain to the supply company would be the same in either case.*

Of course, if all the consumers did the same thing, the gain to the supply authority would be in the same ratio as the reduction in the consumers' demands, *i.e.*, it would be at a decreasing rate the nearer the power factor came to unity; and a tariff based on demand kVA would then be equally fair to both sides. The difference in the economics of the case to the two parties arises from the fact that when the individual improves he brings his demand much nearer to unity, and so decreases the rate of change of the improvement (see p. 252 and Fig. 21), whereas he does not change the lag on the station materially and so does not affect the degree of profitableness to the supply authority. While station power factors remain bad it would therefore be worth while for the authorities to

* It will be noted that if the consumers are not adjacent, but each is situated at the end of a separate transmission line, the gain to the supply company in respect of the improvement in *regulation* would be greater in the first case than in the second, and would in this respect be more in harmony with the rewards given to the consumers.

pay at a level rate for all quadrature kVA injected into the system, or else to charge for kVA at a slightly lower rate to consumers taking it at or nearly at unity power factor.

Summing up this point, it may be said that for a fixed total power factor on the generating station, all wattless leading kVA have the same value, whereas if paid for on the basis of consumers' kVA a bigger saving will accrue to an individual who improves his power factor from bad to medium than one who improves from medium to good. But as regards the gradual improvement of the station power factor as a whole, leading kVA installed later, *i.e.*, as the power factor approaches unity, are of less value to the supply authority in the same way that they show a smaller saving in the consumers' electricity bills.

Bonus/Penalty Tariffs.—Turning now to the second group of tariffs outlined above (Class B), the bill is first computed on a true power and energy basis, and then subject to a series of proportionate allowances or additions dependent on the power factor. In many cases the bonus/penalty is reckoned on the total bill, whereas it was seen above that the bad power factor mainly affects the standing charge and should, therefore, be levied chiefly or only on the demand portion of the bill. When reckoned on the total bill, the consumer with a bad power factor but good load factor will pay heavily, whereas the consumer who has both a bad power and load factor, and, therefore, a small energy bill in comparison with his maximum demand will not be charged so much, and will not be encouraged to install improving plant.

When a bonus only is given for improvement beyond a certain basic figure, a consumer whose power factor is much below this will hardly trouble to install the large amount of plant required to bring the power factor above the basic figure. A further objection lies in the difficulty of metering power factor, and of explaining this to non-technical consumers.

The tariffs of this type which are in operation differ widely both as to the basic power factor employed and the method and amount of the allowances. It is therefore difficult to criticise them as a whole, and each one must be examined on its merits. It may, however, be said that, as a general rule, this type of tariff is less flexible than the other, and does not represent actual costs so well.

Regulation.—Regulation can be defined as the drop in voltage resulting from a load, or the rise when that load is removed, expressed as a percentage of the rated voltage. Thus if a consumer nominally supplied at 100 volts finds that at the time of maximum load the voltage has fallen to 90, the regulation is 10 per cent., and naturally it is extremely important to keep this voltage drop down to as small a figure as possible. The drop may be due to a variety of causes. Thus the prime mover may fall in speed very slightly, and the dynamo will generate slightly less voltage owing both to the speed change and to loss in magnetic field strength. This drop will be further increased by the fall in voltage due to resistances, first in the machine itself, and then in the cables and other gear supplying the consumer. In a large station generating A.C., the voltage drop in the machines themselves can be made extremely slight by means of suitable regulators, and it is chiefly the subsequent losses in the distribution gear which effect the total drop.

In order to illustrate the effect of power factor upon regulation it will be well to take actual figures. In the case of an overhead line whose resistance and reactance drop at full load and unity power factor are 10 per cent. and 20 per cent. respectively, in order to have 100 volts at the receiving end it is necessary to start with the voltage of 112. With the same line and the same kW loading, but at 80 per cent. power factor, it is necessary to start with 125 volts. When the same load is taken at 60 per cent. power factor the starting voltage would have to be 137, so that a boost of 37 volts would be necessary if no power factor improvement were attempted. Thus a bad power factor damages the regulation for two reasons. In the first place it necessarily increases the voltage drop in the cable, since it means a larger current for the same power. Furthermore when the line possesses reactance as well as resistance the extra kVA will actually produce a bigger drop than if the same number of true kW were added, since the addition is a vectorial one. In the above case the currents are in the ratio 1 : 1·25 : 1·67, whereas the voltage drops are in the ratio 1 : 2 : 3.

The question of regulation, whilst it is closely connected with that of power factor improvement, is distinct in many ways. For one thing, regulation is entirely a question for the supply authority, and although it is affected by the character of the load taken, there is no way in which the responsibility

can be passed on to the consumer, except indirectly by such power factor tariffs as have already been considered. Hence if apparatus is being installed solely with a view to improving the regulation, it will naturally be by the supply authority.

The apparatus installed may aim at adding a direct in-phase voltage to that given by the alternator or main transformers, thus compensating for the drop in the lines, or it may aim at phase improvement by means of a quadrature component, thus neutralising the drop due to bad power factors. As regards the second method, any of the apparatus already described may be employed, but it must be remembered that in this case facilities for varying the amount of the phase improvement are very important. In some cases a long line may have an appreciable capacity at light loads (thus causing a *rise* in volts at the receiving end) and a considerable inductance at full load. In such cases a lightly running synchronous machine is very useful, since by suitable field variations it can be made to run either as a reactor or a condenser.

The economic side of regulation is difficult to discuss except in very general terms. Sometimes the voltage *must* be kept within a certain figure at all costs, and, almost always, tolerably good regulation is extremely important, and its value cannot easily be assessed in £ s. d. All that can be said is that in some cases the necessity for better regulation will compel the installation of phase improvement plant, even where it would not otherwise be economically justifiable, and in all cases in which such plant is a sound proposition on other grounds, regulation considerations will provide an additional reason for its installation.

Worked Examples

1. On a 480-volt 50-period circuit a correction capacity of 100 idle kVA is required, and condensers for operation on 600 volts can be purchased for £2 10*s*. per kVA, the output capacity being proportional to the square of the voltage. The choice, therefore, lies between direct connection at 480 volts and operation at 600 volts through an auto-transformer costing with gear £46 12*s*., and having a full load loss of 545 watts. Assuming interest at 6 per cent. and a twenty-year life (zero salvage value) for all apparatus, at what service-price will the two alternatives balance, neglecting any difference between the two condenser losses ?

If condensers are direct connected to 480-volt supply, they must be of such a size as to have a capacity (at 600 volts) of $100 \times \left(\frac{600}{480}\right)^2 = 156\frac{1}{4}$ kVA, costing at £2 10*s*. per kVA £390 12*s*.

Alternatively, a 100 kVA 600-volt condenser costs .	£250 0
To which must be added transformer, etc., at .	£46 12
	£296 12

This alternative is, therefore, cheaper on capital costs by £94, which, at an annual rate of $(0{\cdot}06 + 0{\cdot}0272)$ = £8 4*s*. per annum. But the annual cost of transformer losses = $0{\cdot}545 \times$ service-price, so that the balancing service-price $= \dfrac{8{\cdot}2}{0{\cdot}545} = 15$.

Hence, the transformer equipment is cheaper in capital costs by £94, but if the service-price exceeds 15 (corresponding to energy at $1\frac{1}{2}d$. with 2,400 hours of full load service a year), the direct connection equipment will prove cheaper, since the cost of the transformer losses will then more than balance the capital difference.

2. Induction motors are required to give 20 h.p. at 1,000 r.p.m. for 3,000 hours a year, and the choice lies between ordinary squirrel-cage machines and compensated induction motors employing a commutator. The data of the two types is given below, and the tariff is £5 per kVA of demand plus 0·6*d*. per unit. Determine which of the two types will be the more economical, and by how much per machine per annum.

	Standard S.C. Motor.	Compensated Motor
First cost (per motor) . .	£46	£64
Interest, depreciation and upkeep, reckoned at (per annum) . .	9 per cent,	11 per cent.
Full-load efficiency . . .	87 per cent.	84·5 per cent.
Full-load power factor . .	0·85	Unity

	Standard S.C. Motor.			Compensated Motor.		
True power taken	$\frac{20 \times 0{\cdot}746}{0{\cdot}87} = 17{\cdot}15$ kW			$\frac{20 \times 0{\cdot}746}{0{\cdot}845} = 17{\cdot}66$ kW		
Apparent power taken	$\frac{17{\cdot}15}{0{\cdot}85} = 20{\cdot}19$ kVA					
		£	s.		£	s.
Hence demand charge (kVA. × £5)	=	100	19		88	6
And energy charge (£)—						
$\left(\text{kW} \times \frac{3{,}000 \times 0{\cdot}6}{240} = \text{kW} \times 7{\cdot}5\right)$	=	128	14		132	8
Capital charge—	£46 × 0·09 =	4	3	£64 × 0·11 =	7	1
Total charge	=	£233	16		£227	15

Hence the compensated motor is here the cheaper proposition to the extent of £6 1*s*. per machine per annum, but it will be noted that the squirrel-cage motor performance cited above is somewhat below that which should be obtainable on a good modern machine of this speed.

3. Referring to the previous example, if there were a number of such motors to be installed, would it be cheaper to install plain squirrel-cage motors and correct to unity power factor by static condensers, assuming that these cost £3 per kVA with an annual allowance of 14 per cent.* and have losses of 0·005 kW per kVA ?

In order to use definite figures, suppose there are ten motors to be installed. Quadrature component required in order to correct to unity p.f. $= \sqrt{201{\cdot}9^2 - 171{\cdot}5^2} = 106{\cdot}6$.

This will require a 107 kVA condenser having 0·535 kW loss.

Capital charge for condenser	= £3 × 107 × 0·14 =	£44 19
Losses charge for condenser	= £ 7·5 × 0·535 =	£4 0
		£48 19
Saving in demand charge	= £5 × (201·9 − 171·5)	£152 0
	Net saving =	£103 1

i.e., £10 6*s*. per motor.

Hence the above arrangement is cheaper by £10 6*s*. per annum per motor than the plain motor, or cheaper by £4 5*s*. than the compensated motor. It would, of course, be still more economical to

* It will be noticed that as this is not a choice between alternative machines, but involves an *additional* piece of apparatus, it is necessary to allow for items such as the extra space required and consequent housing expenses.

correct by condenser, not to unity power factor, but to the angle whose sine is $\frac{£5}{£3 \times \cancel{0{\cdot}14}} = \cancel{0{\cdot}119}$ 0·074, *i.e.*, 7 degrees (p.f. 99·3 per cent.). See next chapter.

4. On a certain system it is desired to obtain power factor improvement to the extent of 270 (quadrature) kVA, and at the same time a drive of 90 h.p. The following alternatives are available: determine which will be the more economical for a service-price of 5, assuming an annual allowance in each case of 10 per cent. of the capital cost—

280 kVA synchronous motor, costing £650, and having losses of 15 kW.

300 kVA static condenser combined with 90 h.p. induction motor, the two together costing £950, and having total losses of 6 kW.

Extra cost of induction motor equipment = £300 at 10 per cent. = £30 per annum.

Extra cost of synchronous motor losses — 9 kW at service-price of 5 = £45 per annum.

Hence the synchronous equipment costs £15 per annum more than the other; and in order to balance, the service-price would have to be as low as 3·3—corresponding to energy at $\frac{1}{3}d.$ with full-load service for only 8 hours × 300 days a year.

5. A consumer working at 80 per cent. power factor has a true power demand of 1,000 kW, and his electricity bill is £10,000 a year. He is offered a rebate of $\frac{1}{3}$ per cent. for every 1 per cent. improvement in power factor. Determine the saving (if any) which can be made by the improvement of his power factor up to 95 per cent., using for this purpose static condensers of first cost £3 10*s.* per kVA, annual allowances (to cover all capital and overhead expenses) 16 per cent., and negligible losses.

At 80 per cent. p.f. the wattless component

$$= \sqrt{\left(\frac{1{,}000}{0{\cdot}8}\right)^2 - 1{,}000^2} = 750 \text{ kVA}$$

At 95 per cent. p.f. the wattless component

$$= \sqrt{\left(\frac{1{,}000}{0{\cdot}95}\right)^2 - 1{,}000^2} = 328{\cdot}6 \text{ ,,}$$

Hence corrective capacity required = 421·4 ,,

Annual cost of condenser equipment = £3 10s. $\times$ 0·16 $\times$ 421·4 = £236

But rebate obtained for 15 per cent. improvement

$$= £10{,}000 \times \frac{\frac{1}{3}}{100} \times 15 \qquad = £500$$

Hence the rebate is more than twice the cost of obtaining it, leaving a total net saving of £264 per annum.

CHAPTER XIV

POWER FACTOR (*continued*)

CHOICE OF STATIC AND SYNCHRONOUS CONDENSERS

Improvement Calculations.—In the present chapter it is proposed to consider the economics of power factor improvement when this is carried out by means of special plant installed for that sole purpose. It will be assumed that the size of such plant is measured by its kVA capacity, and that this kVA is completely at quadrature—the current leading the pressure by 90 degrees. This could only be strictly true if the plant had no losses whatever, but in practice the losses are small enough to make the assumption a very close one. The most important calculation which has to be made is to find how much the size of the generating plant is reduced for any given addition of improvement plant.

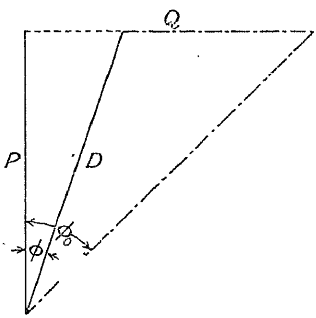

FIG. 20.—Power Factor Improvement.

The magnitude of the load probably varies from minute to minute, but it may be assumed that the power factor remains tolerably constant, and in this case the calculations should concern the instant of maximum load. If P is the true power taken at this instant and ϕ the angle of lag, the maximum demand will consist of $\frac{P}{\cos\phi}$ kVA apparent power (Fig. 20). If the power factor be improved slightly, lessening the lag by a small angle $\delta\phi$, the reduction in demand will be

$$P\left\{\frac{1}{\cos\phi} - \frac{1}{\cos(\phi-\delta\phi)}\right\} \text{kVA}$$

In order to effect this improvement, the size of plant required

(measured in quadrature kVA) will be $P\,[\tan\phi - \tan(\phi - \delta\phi)]$ Hence the ratio

$$\frac{\text{kVA reduction in demand}}{\text{Quadrature kVA required}} = \frac{\dfrac{1}{\cos\phi} - \dfrac{1}{\cos(\phi - \delta\phi)}}{\tan\phi - \tan(\phi - \delta\phi)}$$

and it can be shown that when $\delta\phi$ is made very small this equals $\sin\phi$.

The above can be better expressed as follows, using the calculus. Suppose that improvement has been carried out from an initial lag ϕ_0 to the angle ϕ, and let the demand and the quadrature kVA be represented by D and Q respectively (Fig. 20).

$$\text{Then } D = \frac{\mathrm{P}}{\cos\phi}, \text{ and } Q = P\,(\tan\phi_0 - \tan\phi).$$

When ϕ is varied slightly, the rate of change of D with ϕ

$$\frac{dD}{d\phi} = \frac{dP}{d\phi\cos\phi} = P\frac{\sin\phi}{\cos^2\phi}.$$

And the rate of change of Q with ϕ

$$\frac{dQ}{d\phi} = \frac{d}{d\phi}P\,(\tan\phi_0 - \tan\phi) = -P\sec^2\phi \text{ (since } \phi_0 \text{ is constant).}$$

Hence the rate of change of D with Q,

$$\frac{dD}{dQ} = P\frac{\sin\phi}{\cos^2\phi} \times \frac{1}{-P\sec^2\phi} = -\sin\phi.$$

In words, D goes down as Q goes up, and always proportional to the sine of the particular angle of lag reached at that instant. When the lag is 90 degrees (zero power factor) 1 kVA of quadrature plant will save 1 kVA of generating plant; when the angle is 30 degrees (86·6 per cent. power factor) the same quadrature plant will only save half as much. It will be noted that the rate of change is independent of the value of the initial lag ϕ_0 and also of the magnitude of the power P.

In Fig. 21 the values of $\sin\phi$ are plotted to a base of ϕ, and on the same base are marked the values of the power factor, $\cos\phi$. The curve therefore shows the rate of change of D with Q, *i.e.*, the saving in kVA of generating plant effected by the addition of 1 (quadrature) kVA of improving plant at any particular position of the power factor. Thus at 70 per cent. power factor ($45\frac{1}{2}$ degrees lag) 1 quadrature kVA will save 0·714 kVA of supply plant, at $86\frac{1}{2}$ per cent. power factor the

figure is 0·5, at $99\frac{1}{2}$ per cent. it is 0·1, getting less the nearer the power factor approaches unity.

An important point to notice is that this only expresses the *instantaneous* rate of change, assuming some one particular value for the angle of lag, so that it is not a reliable guide to the total amount of kVA saving unless the change is small compared with the total load. Thus to a supply authority working at 70 per cent. power factor, 1 quadrature kVA will actually save the figure mentioned, namely, 0·714 kVA, but to an individual small consumer having a true power demand

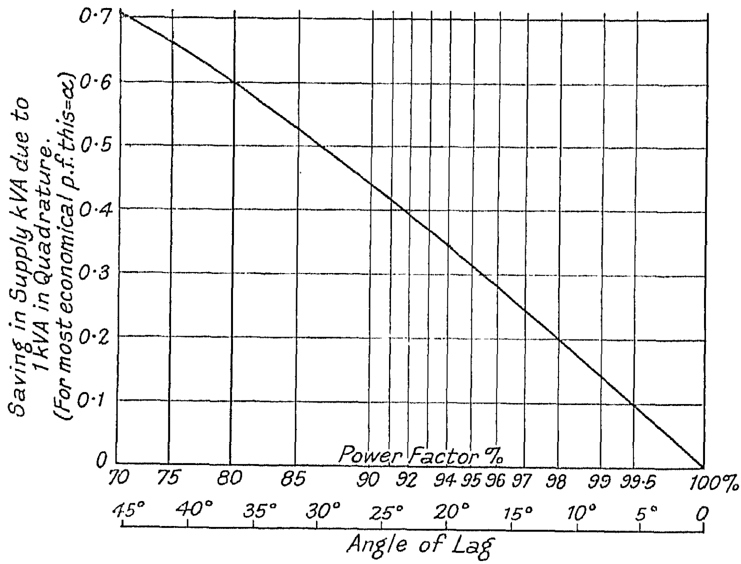

FIG. 21.—Rate of Change of kVA's.

of, say, 10 kW and an initial power factor of 70 per cent., the introduction of 1 quadrature kVA will only reduce his kVA demand by 0·696, because in the latter case the result of the installation will be materially to change his angle of lag, and so the ratio existing between the two rates of change. This difference in viewpoint as between consumer and supply authority has already been touched upon (p. 242).

It will be noticed that the basis of Fig. 21, as of the next one in the book, is an even scale of degrees (angle of lag), upon which is also shown the percentage power factor. There are several advantages in adopting this scale in preference to

any other. In the first place, power factors nearer unity are spaced further apart, so that the conditions in this region are shown on a magnified scale. Another advantage is that for small lags tan ϕ is approximately proportional to ϕ, so that equal distances along the base represent roughly equal amounts of improvement as measured by the kVA of the plant installed.

Choice of Size (Standing Charge Basis).—In the following calculations only one type of situation will be considered—that in which plant is installed purely for the purpose of power factor improvement. This rules out all cases in which the plant serves also as a driving or converting unit. Within this situation only two problems will be considered—choice of size and choice of type, both being made on a purely economic basis. There is no logical order in which to consider the two questions, since either may be said to involve the solution of the other. Strictly speaking the best size can only be determined when the type (and therefore price per kVA) is known, and, on the other hand, the most economical type depends on the size. In this case the size problem will be considered first, since this is the one which is more generally met with.

Since a bad power factor has two effects, loss of kVA capacity and loss of energy, the economic size of plant can be determined either on a basis of standing charges, based on maximum demand, or on a basis of working charges, based on actual consumption, or both. It has been seen, however, that the extra kVA of demand cost in the neighbourhood of three-quarters as much per unit as the true kW, and when metered are usually charged for at the full rate, whereas the extra kVA hours of consumption only cost about a quarter as much per unit as the true energy, and are usually not metered or charged for at all. Hence, whether the problem is looked at from the point of view of reducing the electricity bill or of reducing the real costs of supply, it is the loss in capacity to meet demand which is much the more important effect, and at first only this will be considered.

The standing costs of supply, *i.e.*, those expenses proportional to maximum demand, and usually comprising every cost except fuel, will be partly dependent upon true power (*e.g.*, the steam plant) and partly upon apparent power (*e.g.*, the electrical plant and most of the upkeep). Using the symbols P, Q and D as before for the power kW, quadrature kVA and

demand kVA respectively at the instant of maximum demand, the costs of supply can be represented by $XP + YD$, where X and Y are constants, both being reckoned on the same basis of annual costs. In the same way the expenses of the improvement plant can be expressed as ZQ, where Z is the annual cost per kVA of the plant employed for this purpose.

Expressing these in terms of ϕ, the total cost becomes $XP + Y \dfrac{P}{\cos\phi} + ZP\ (\tan\phi_0 - \tan\phi)$. When ϕ is varied the rate of change of the cost with respect to ϕ will be given by $YP \dfrac{\sin\phi}{\cos^2\phi} - ZP \sec^2\phi$. (It will be noted that the two rates of change are exactly as before, but multiplied by the constants Y and Z. The first term, being entirely constant, disappears.) The minimum total cost will occur when these two rates of change are equal and opposite, *i.e.*, when

$$YP \frac{\sin\phi}{\cos^2\phi} = ZP \sec^2\phi \text{ or } \frac{Z}{Y} = \sin\phi = \sqrt{1 - (p.f.)^2}.$$

It will be seen that the result is entirely independent of ϕ_0, P and X.

The ratio Z/Y, which may be denoted by the symbol α, is easily calculated from the data of the problem, and hence the economic limit of power factor correction can be calculated. Thus if the first cost of the generating plant be £40 per kVA, and that of the improving plant £3 per kVA, and if the interest, depreciation and all upkeep expenses varying with capacity be represented by 15 per cent. per annum in the case of the former and 12 per cent. in the latter, the ratio α will be $\dfrac{3 \times 0{\cdot}12}{40 \times 0{\cdot}15} = 0{\cdot}06$, and the most profitable point to which to carry correction will be that in which $\sin\phi = 0{\cdot}06$, *i.e.*, $\phi = 3\frac{1}{2}$ degrees, or $\cos\phi = 99{\cdot}8$ per cent.

Referring again to Fig. 21, the ordinates of this curve can be taken as representing α, so that if the relative costs are known, and α is calculated, the curve will at once show the angle and power factor to which improvement can profitably be carried. Whatever the initial conditions, if $\alpha = 0{\cdot}1$ the most economical power factor is 99·5 per cent., if $\alpha = 0{\cdot}5$ it is 86·6 per cent., and so on.

Value of α.—The above formula was first given in the paper on "Power Factor Improvement," by the late Gisbert Kapp (*Journal I.E.E.*, Vol. 61, January, 1923), which may be said to have laid the basis of the economic study of the problem. It will be seen that the question of the economic choice of size turns entirely on the value of α, where α is the ratio: cost per kVA of improvement plant divided by cost per kVA of supply plant. The figures mentioned by Dr. Kapp for the usual range in practice of α were from 0·3 to 0·1, the value of 0·25 being suggested as typical for a static condenser installation. There are, however, several causes which may lead to a modification of these figures in the direction of somewhat lower ones. It must be remembered for one thing that power factor improvement is a much newer thing than generation, and there is reason to anticipate that when its practice becomes more general the cost of the plant will fall in comparison with that of generation and distribution gear.

A further point to notice in this connection is that the above figures were originally put forward as representing the ratio of the two capital costs, whereas it is the annual costs which must be compared. The standing cost of a private generating plant is not merely the cost of the interest and depreciation on the capital employed, but it must include every item varying with the kVA and independent of the consumption, *i.e.*, the annual cost of the buildings and foundations, all rents and taxes levied on them, maintenance, upkeep and management. The same kVA of improvement plant not only costs less to buy, but probably needs less floor space, lighter foundations, smaller rates, etc.; and this will be not only less per kVA, but probably less also per £1 of capital expenditure. The same remark applies to the upkeep and maintenance, particularly if the plant is static; even the depreciation may be less owing to a longer working life.

Taking the figures suggested by Dr. Kapp for first cost of generating and improving plant, namely, £12 and £3 per kVA respectively, these as they stand give a ratio of $\frac{1}{4}$ for α, and a corresponding economical angle of lag (to which the system should be improved) of $14\frac{1}{2}$ degrees, *i.e.*, a power factor of 96·8 per cent. But if the annual percentages necessary to cover the various items detailed above were 15 per cent. and 12 per cent. respectively, the value of α would be $\frac{3 \times 0{\cdot}12}{12 \times 0{\cdot}15}$ $= 0{\cdot}2$, and the economical angle of lag would be $11\frac{1}{2}$ degrees

(p.f. 98 per cent.). Hence even this small difference between the two percentages will make an appreciable change in the size of the phase advancing plant which it would be economical to install.

Naturally when some of the items mentioned above, such as floor space, attendance, etc., are already available or would be used to the same degree by either type of plant, it is quite legitimate to leave such items out, and let the annual percentage cover only the pure interest and depreciation. Thus if the interest were at 6 per cent. throughout, and the useful lives of the generating and improving plant were assessed at twenty and twenty-five years respectively, the total percentages would be 8·7 and 7·8, which would only modify the value of α from 0·25 to 0·224, with a correspondingly small change in the economic degree of improvement.

The conversion from first costs to annual costs is particularly important when the authority installing the improving plant is not the authority supplying the electricity. In such cases every possible extra expenditure entailed by the proposed installation, including such things as added responsibility and worry, must be scrupulously assessed and included in the percentage before the cost of such plant is compared with that of the supply kVA. Hence a higher percentage will frequently have to be employed than when comparing the installation of alternative types of plant, and an allowance of at least 12 to 15 per cent. per annum of the capital cost will usually be necessary to cover all these items, in addition to interest and depreciation, the lower figure applying more particularly to static condensers.

As an example of the above, let it be supposed that the consumer is charged £5 per annum per kVA of demand, and improving plant costs £3 per kVA with an annual allowance (to cover all the above items) of 15 per cent. The value of α will then be $3 \times 0{\cdot}15/5 = 0{\cdot}09$, giving an economical lag of 5 degrees and power factor 99·6 per cent. To take another case, if the supply charge were £8 per kVA and the annual allowances were 12 per cent., the value of α would then be just half the above figure.

On the other hand, when the choice is based on a comparison of capital costs, and when the supply figure is that for the whole station, not all of this will necessarily be proportional to kVA. Thus if the capital cost is £12 per kW at unity power factor, hen, employing the figure of three-quarters suggested in the

last chapter, this could be expressed as £3 per kW plus £9 per kVA or £12 per kW plus £9 per extra kVA. The figure to be employed in calculating the most economical lag will then be only £9, and this will have the effect of raising α. Moreover, the figure of three-quarters was intended to cover a complete supply system, in which the transmission and distribution costs were about the same as the generation costs. In a private generating plant this would not be so, and the proportion might be considerably less than three-quarters.

In order to narrow the question, no mention has been made of cases in which the power factor improvement is combined with a driving or converting unit, although the above calculations will apply equally well to such a case. The cost of improvement will then be considerably less, possibly only 10s. per idle kVA in the case of a phase advancer fitted to a large induction motor, and this again will have the effect of making α lower still. Summing up, it may be said that as regards plant employed for the sole purpose of power factor improvement, α may usually be expected to lie between 0·2 and 0·1, with outside limits of, say, 0·3 and 0·05. When the supply costs proportional to kVA are very low it may be higher than this, and it would probably be lower than this in the case of phase advancers on large driving motors.

Total Saving.—It is a common practice in power factor questions to work out the gross saving effected by the installation of improvement plant, and to compare this with the expenditure required, thus showing or attempting to show that a satisfactory return can be obtained on the necessary capital. This is apt to be misleading as a guide to how far to carry improvement, since even if this were carried too far there might still be a handsome return on the total expenditure. The economic limit of power factor improvement will be reached not when the rate of profit on the outlay reaches a maximum and begins to go down, but when the gain which accrues from the last small increment of improvement expenditure only just pays for itself. This is comparable with the "marginal dose" in the cultivation of land (see p. 78), and it will occur, as shown in the above calculation, when the rates of change of expenditure on supply and on improvement are equal and opposite.

If it is desired to know the net saving which has accrued in any given case, this can easily be calculated if the angles of lag

are known before and after the improvement was effected. If improvement is carried out from an initial lag ϕ_0 to a smaller lag ϕ, then, using the above symbols and omitting any costs not proportional to kVA,

$$\frac{\text{Net saving}}{\text{Original cost}} = \frac{PY\left(\frac{1}{\cos\phi_0} - \frac{1}{\cos\phi}\right) - PZ(\tan\phi_0 - \tan\phi)}{PY\left(\frac{1}{\cos\phi_0}\right)}$$

$$= \frac{\frac{1}{\cos\phi_0} - \frac{1}{\cos\phi}}{\frac{1}{\cos\phi_0}} - \frac{Z}{Y}\cos\phi_0(\tan\phi_0 - \tan\phi)$$

$$= 1 - \frac{\cos\phi_0}{\cos\phi} - \frac{Z}{Y}(\sin\phi_0 - \cos\phi_0\tan\phi)$$

If improvement is carried out up to the economic limit, then $\sin\phi = \frac{Z}{Y} = \alpha$, so that $\cos\phi = \sqrt{1-\alpha^2}$, and $\tan\phi = \frac{\alpha}{\sqrt{1-\alpha^2}}$. The above ratio then becomes

$$1 - \frac{\cos\phi_0}{\sqrt{1-\alpha^2}} - \alpha\left(\sin\phi_0 - \cos\phi_0\frac{\alpha}{\sqrt{1-\alpha^2}}\right)$$

$$= 1 - \frac{\cos\phi_0}{\sqrt{1-\alpha^2}}(1-\alpha^2) - \alpha\sin\phi_0 = 1 - \sqrt{1-\alpha^2}\cos\phi_0 - \alpha\sin\phi_0$$

$$= 1 - (\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0) = 1 - \cos(\phi_0 - \phi)$$

Hence, whatever is the expenditure proportional to kVA at any instant, an improvement of the angle up to the economic point by a definite number of degrees will always produce a definite percentage reduction on this expenditure, namely, 1·5 per cent. reduction for 10 degrees improvement, 13·4 per cent. for 30 degrees, and so on. Thus when a consumer paying on a kVA basis installs improving gear, his net proportionate saving on his previous expenditure will be given by 1 minus the cosine of the angle through which he improves, provided he improves just up to the economic limit. As an example, a consumer whose demand bill was originally £100 a year and power factor 70 per cent. (45½ degrees lag) finds that for quadrature gear suited to his case $\alpha = 0{\cdot}2$, and so he improves up to that angle whose sine is 0·2 (11½ degrees)—a total improvement of 34 degrees. His saving will then be £100 (1 — cos 34 degrees) = £17 2*s*. per annum.

Figure 22 shows the percentage saving for five different values of α when improvement is carried out to the economic point. The base of the curve is the same as that of the previous figure, but extending twice as far, and in this case it represents the *initial* lag and power factor, before improvement was effected. The curve then shows the biggest saving which can be made in any given case. It will be seen that each curve

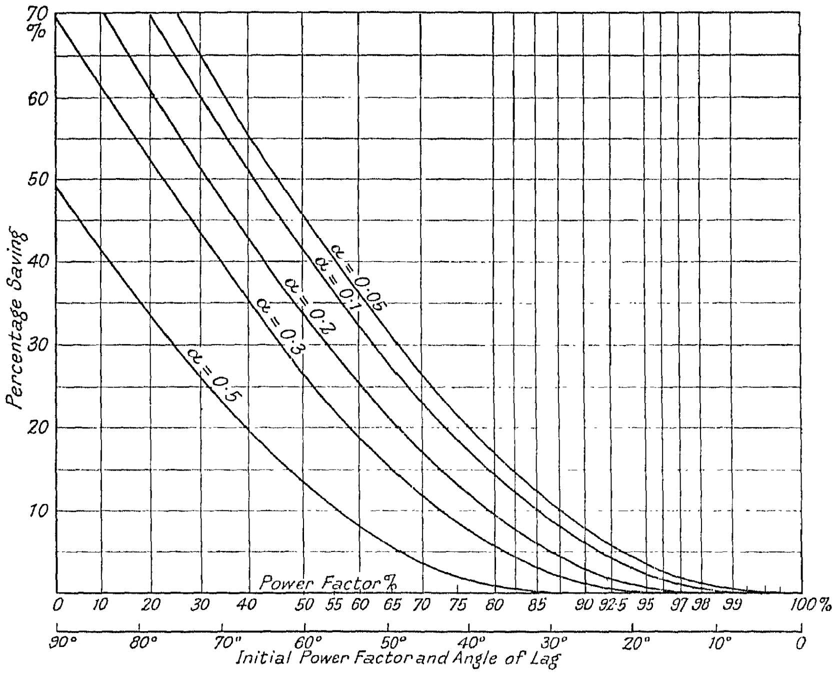

FIG. 22.—Percentage Saving.

falls to zero at the angle of lag indicated for that value of α in Fig. 21.

It must be made clear that the percentage shown is not the return on the capital invested ; it is the actual net saving on the transaction after subtracting the cost of making the improvement, and it is expressed as a percentage of the original cost, provided the latter was all proportional to kVA. If all the costs are expressed on a yearly basis, this percentage will give the annual saving, whereas if all are expressed in capital costs it will give the lump sum saving.

Thus in the case mentioned above, where the original installation was costing £100 a year in items proportional to kVA, and in which improvement could be carried out on terms represented by the value $\alpha = 0{\cdot}2$, the saving which could be effected is £34 a year if the initial power factor is 50 per cent., £17 a year if it is 70 per cent., and so on. It will be seen that with low initial power factors the proportionate saving is a very considerable one, and however small the initial lag (provided it is greater than the angle whose sine is α) there is always some profit to be made in improving it up to this value.

Graphical Representation.—The points illustrated in the last two figures, namely, the economic angle and the percentage net saving, are the most useful figures to have for the purpose of power factor calculations, but they cannot be said to convey a very clear picture of what is actually taking place. Figure 23, which is drawn to represent any set of conditions having a particular value of α (in this case $0{\cdot}2$), is an endeavour to supply this need. The construction is as follows: OP represents the standing or demand cost of the supply if taken at unity power factor, all items not varying with kVA having been omitted. PQR is at right angles to OP, so that any line such as OQ or OR lagging behind OP will represent to the same scale the cost of the supply if taken at such an angle. OQ represents the direction of the demand after improvement has been carried out up to the economic point, *i.e.*, $\sin \angle POQ = \alpha = 0{\cdot}2$ in this case. QD is an arc of a circle with O as centre, and ABC is a line at right angles to OP and situated so that $AO/OP = \alpha$.

If the chain-dotted line $OCDR$ represents the maximum demand before improvement is carried out, on any system in which $\alpha = 0{\cdot}2$ ($\angle POR = \phi_0$ and $OP/OR =$ initial power factor), its length will represent the cost of this demand to the scale chosen. If improvement be carried out to the most economical point, the saving in supply cost will be represented to the same scale by $OR - OQ$, *i.e.*, by the vector DR. In order to achieve this it is necessary to install plant giving a quadrature kVA represented by QR, but as this is cheaper per unit than the supply kVA in the ratio α, its cost will be represented not by QR, but by BC. Thus whatever the direction of the initial demand, if a straight-edge is laid across or a line drawn through O in this direction, its intersections of the shaded and etched portions of this diagram will show at once both the saving and cost which would result from power factor improvement.

It will be seen that *DR* is always greater than *BC*, *i.e.*, there is always some net saving, but of course this gets smaller the less the amount of improvement there is to be effected. A further point to notice is that, unlike the two previous diagrams, this one does not depend for its applicability upon improvement being carried out up to the economic point. Thus if improvement is carried out from an angle of lag represented by *OCDR* to a lesser angle represented, say, by *OC'D'R'*, the saving in supply cost will be represented by $DR - D'R'$ at a cost (to the same scale) of *CC'*.

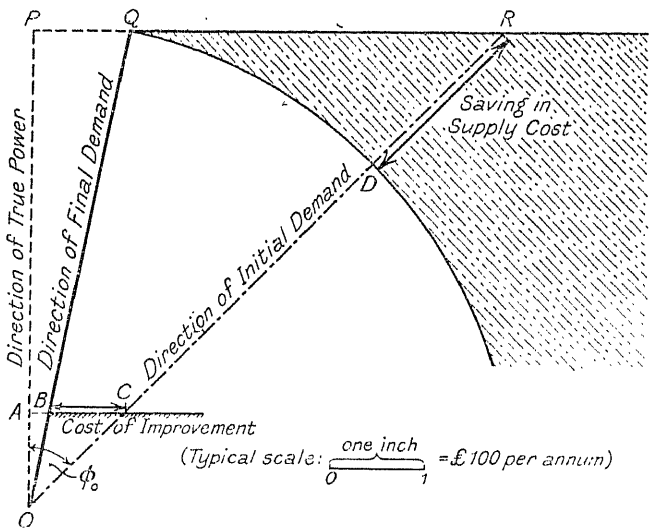

FIG. 23.—Saving and Cost of Power Factor Improvement.

The scale of the diagram is in £, which may be either annual or capital. Thus if the standing cost of supply (or that portion of it proportional to apparent power) is £5 per annum per kVA of demand, and that of power factor improvement is £1 per annum per quadrature kVA, and if the true power is 100 kW, the line *OP* will represent £500 per annum, and the money scale would be as indicated at the bottom of the diagram, *i.e.* 1 inch = £100 per annum. If the standing supply cost (proportional to kVA) is expressed as a capital value of £15 per kVA, and that of the improvement plant as £3 per kVA, and if these capital costs are directly comparable with no difference in their annual percentages, then for 100 kW load *OP* would

represent £1,500, and the scale would be 1 inch = £300. The diagram is useful not so much to save detailed calculations as to give a graphical impression of the gross saving and the extra cost, and of the proportions which these bear to the original and final expenditures.

Working Charge Modification.—So far the decision has been made purely on the basis of standing charges rated on maximum demand. But the introduction of phase improving plant will, in general, lower also the working charges, since it will save I^2R losses in cables and generating plant to a greater extent than it adds losses of its own. This, of course, applies more particularly when static condensers are employed, and in this country it is chiefly of interest when the supply authorities are installing the improving plant, since although the extra kVA hours due to bad power factor cost money, the extra is not usually passed on to the consumer in the form of a tariff.

There are several ways in which the cost of the losses can be brought in, as, for example, when the losses saved by phase improving plant are balanced against the cost of the plant itself, but this excludes entirely the increased capacity achieved and its effect upon the standing charge. Such a calculation will usually indicate an economical angle of lag considerably larger than that worked out on a basis of standing charges only. What is needed, therefore, is to combine both considerations; this will result, of course, in a smaller economical angle than that given by either of them separately.

By assigning symbols to all the different variables it is easy to build up a formula taking account of both sets of costs (see footnote, p. 264), but in the author's opinion this is too complicated to be of much practical service. In such a case it is clearer, and, on the whole, more useful to take actual figures, and in what follows a single typical example will be considered and the results plotted graphically. This enables the differentiation to be carried out by inspection, and it also shows the positions not only of lowest total cost, but also of the lowest of each of the two components.

As in many other cases of economic choice, two of the items which vary most, namely the load factor or hours of service per annum, and the price of energy, are important only in their product, and following the procedure of Part II., the term service-price will be used to denote the product: hours of service per annum multiplied by cost of energy (£). In the

present case a service-price of 3 will be assumed, so that every kW of load will cost £3 a year. (A low value is taken, since in this case it corresponds to the energy portion only of a two-part tariff.) In order to simplify the problem still further it will be assumed that the service is at full load during all the hours in question, and that the phase advancing is carried out by means of static condensers having a loss of $\frac{1}{2}$ kW per 100 kVA capacity, the condensers being disconnected except during the hours of load.

The standing charge of supply to cover all the capital and other expenses proportional to kVA can be taken as three-quarters of the cost per kW, or taken directly from the demand tariff if this is on a kVA basis; and it is here assumed to be £5 per annum per kVA. The standing charge for the phase improving plant is taken as £1 per annum per quadrature kVA ($i.e.,\ \alpha = \frac{£1}{£5} = 0{\cdot}2$). The working charge per kVA hour can be taken as a quarter of that per kW hour, so that each extra kVA of load will cost—service-price/4 = £$\frac{3}{4}$ per annum, whilst each kW of load costs £3 per annum.

The best way of computing the various costs is to consider a load of 1 kW, and starting at unity power factor, this latter being achieved by the employment of phase improving machinery. The standing cost of such a load is £5 per annum plus the cost of the improvement plant. Now let the same power be taken, but lagging by an angle ϕ. The extra supply cost on account of the increased demand $= £5\left(\frac{1}{\cos\phi} - 1\right)$, the saving in the capacity of the improvement plant is $\tan\phi$ kVA, and as this costs £1 per annum per kVA, the net extra cost resulting from the lag of ϕ is given by $5\left(\frac{1}{\cos\phi} - 1\right) - \tan\phi$ £ per annum.

Consider now the working costs for the hours of service and price of energy represented by a service-price of 3. At unity power factor the working costs for 1 kW of load are £3 plus the cost of the losses in the phase improvement plant employed. When the same power is taken, lagging by an angle ϕ, the supply cost is increased owing to the extra transmission, etc., losses, and decreased owing to the smaller phase improvement plant. The former is taken as a quarter of the kVA increase, and the

latter is found from $\frac{1}{2}$ per cent. of the quadrature kVA reduction, both items being multiplied by the service-price of 3 to bring them to £ per annum. Hence the increase in working costs due to the above lag will be $\frac{3}{4}\left(\frac{1}{\cos\phi} - 1\right)$ £ per annum, and the decrease in the cost of the improvement plant losses will be $3 \times \frac{\frac{1}{2}}{100} \tan\phi$ £ per annum.

The net total costs for 1 kW at an angle ϕ will therefore be the old total costs (at unity power factor) plus

$$5 \times \left(\frac{1}{\cos\phi} - 1\right) - 1 \times \tan\phi + \tfrac{3}{4}\left(\frac{1}{\cos\phi} - 1\right) - 3 \times 0{\cdot}005 \tan\phi \text{ £}$$

per annum. It is easy to see the origin of each of the above figures and to change them to suit other data, since the 5 and the 1 multipliers refer to the costs of supply and improvement plant respectively, the 0·005 refers to the phase improvement losses, and the 3 and 4 refer respectively to the service-price and to the relative cost per extra kVA hour. Thus if the losses in the improvement plant can be neglected (as they frequently can) the item 0·005 disappears. On the other hand, if the authority putting in the plant is charged nothing for extra kVA-hours, the figure 4 becomes infinity, *i.e.*, the $\frac{3}{4}$ disappears.*

In the curve, Fig. 24, the ordinates represent the costs in £s. per annum involved in bringing the final power factor and angle of lag to the figures shown on the base. The ordinates do not, however, all start from the same zero point, and in some cases this zero point cannot be identified. Thus the cost of the improvement plant and the saving it effects will depend on the initial power factor, whereas it is desirable to have a graph which will be equally applicable whatever the initial conditions.

* If preferred it can be expressed in general terms as follows :—
Extra cost per kW of load due to angle of lag ϕ

$$= (C + \beta SP)\left(\frac{1}{\cos\phi} - 1\right) \quad (\alpha C + QSP) \tan\phi$$

where C = supply cost per kVA β = Ratio $\frac{\text{cost per extra kVAh.}}{\text{cost per true kWh.}}$

SP = service-price. Q = ratio $\frac{\text{losses in kW}}{\text{output in kVA}}$ of improvement plant, and α has the same value as before.

Fortunately the economic position depends only upon the slopes of the various lines, not upon their heights, so that this requirement is easily satisfied. It will be seen that there are three

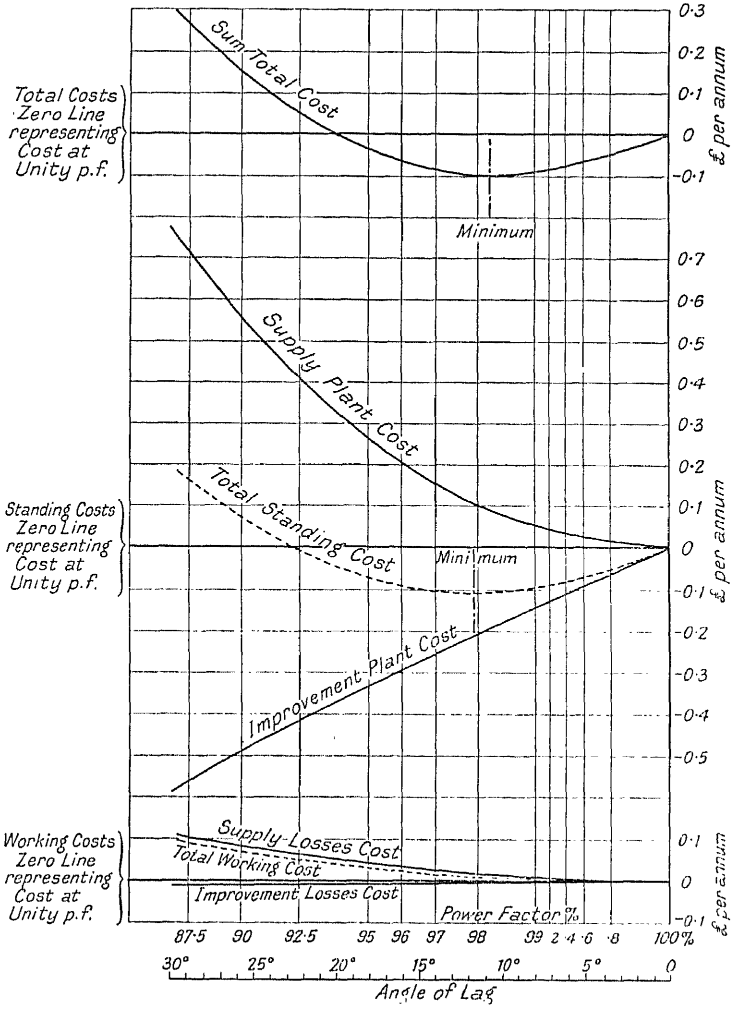

FIG. 24.—Economical Power Factor. (All costs.)

zero lines on the curve, representing the cost at unity power factor of the working, standing, and total expenses respectively. The positions of these zero lines on the curve have been selected

quite arbitrarily so as to avoid clashing, and in such a way as to show clearly the various slopes. It will be seen that the standing costs are a minimum for an angle of lag of $11\frac{1}{2}$ degrees ($\sin\phi = 0{\cdot}2$), whilst the working costs are a minimum at zero lag. Adding the two gives a total cost which is a minimum at a slightly smaller lag than the standing charge one—in this case $10\frac{1}{2}$ degrees.

It will be noted that the modification due to considering the effect of losses in addition to the demand costs is an extremely slight one. Had a lower figure been chosen for the cost of condensers than £1 per annum (which was selected to give a value of $\alpha = 0{\cdot}2$ without too low a figure for the supply cost) the modification due to considering the losses would have been little or no greater. It is, therefore, evident that even when static plant is being employed, the effect of losses on the economic degree of power factor improvement can usually be neglected. When synchronous plant is employed this effect is still less important, since such plant will have appreciable losses of its own and these will tend to neutralise any saving in the supply losses.

Choice of Type.—As in the other examples of this chapter the only choice considered below is that of apparatus installed solely for power factor improvement, the alternatives in question being the static and the synchronous condenser. Since the former is built up of small units its price per kVA is approximately constant, whereas the synchronous condenser resembles most other electrical apparatus in costing less per kVA the larger the size. In general terms, then, it may be said that the static alternative will have the advantage on smaller sizes and the synchronous one on larger, the question being to determine at which point the balance can be struck. For corrective capacities below 100 to 200 kVA the static condenser is both cheaper and more efficient, and will necessarily be chosen unless there are other considerations: such cases will therefore not be discussed here. On larger sizes the synchronous machine has a lower first cost, but greater losses and upkeep; in any given situation the one can be balanced against the other, and an economic choice made.

There are necessarily many points of qualitative difference which cannot easily be expressed in economic terms and so brought within the scheme of comparison. For some purposes,

e.g., in situations where there is at present little or no supervision, the static plant would have definite advantages. On the other hand, where a considerable variation in the corrective capacity is needed (*e.g.*, for regulation purposes) the synchronous plant would be preferred. In so far as the differences lie in cost of maintenance and supervision, shortness of life or poorness of scrap value in the event of changes being needed before the end of life, they appear to be chiefly in favour of the static condenser; and in the general example worked out below an attempt is made to express these differences economically in the form of different annual percentages (12 per cent. and 15 per cent.).* At the same time it must be remembered that a decision on economic grounds cannot be considered exhaustive unless *all* the differences have been quantitatively expressed; and in other cases the economic determination can only supplement, it cannot supplant, that made on technical grounds.

In the following calculation it is assumed that the service is fixed and known beforehand; namely, the magnitude of the correction required, the hours of service per annum and the cost of energy. Since there is to be no difference between the two alternatives as regards quadrature component, the only supply cost to be considered is the energy cost due to the losses in the plant installed, which will be assumed to be charged at a flat rate per unit. The difference in the two energy costs (which will depend not only upon the two efficiencies, but also upon the conditions of service) must then be compared with the difference in first cost and maintenance of the two types of plant.

Indicating the service-price by the initials *SP*, the following symbols will also be employed :—

C = first cost (£) per quadrature kVA of the plant employed.
r = annual rate necessary to cover interest, depreciation, maintenance, rates, and all other expenses incurred or increased as a result of the installation.
Q = loss ratio = losses in kW divided by output in kVA.

(The suffixes t and y indicate the static and synchronous installations respectively.)

* In an interesting comparison between the two types in *The Electrician*, of August 28th, 1925, S.Q.Hayes states that below 300 kVA (on 2,300 V.) or 150 kVA (when transformers are used), the static condenser is lighter, whilst below 100 kVA it takes less floor space. Above these sizes the synchronous machine has the advantage in the directions named.

At the critical point, when the two alternative installations cost the same, $C_t r_t + Q_t SP$ must equal $C_y r_y + Q_y SP.$, both being in £ per annum per quadrature kVA; or $C_t r_t - C_y r_y = SP\,(Q_y - Q_t)$. In any given case if the first group of costs (*i.e.*, the terms in t) are greater, the static alternative will prove the more expensive, and *vice versâ*.

To take a single example, if the static and synchronous plants cost £3 and £1 per kVA, have losses of 0·5 and 3 kW per 100 kVA, and have annual percentages of 12 and 15 respectively, the service-price at which the two will balance will be given by

$$\frac{C_t r_t - C_y r_y}{Q_y - Q_t} = \frac{3 \times 0{\cdot}12 - 1 \times 0{\cdot}15}{0{\cdot}03 - 0{\cdot}005} = \frac{0{\cdot}36 - 0{\cdot}15}{0{\cdot}025} = 8{\cdot}4.$$

Hence for any service-price less than 8·4, synchronous plant would prove the cheaper, but if the service-price were above this, static plant should be installed. It will be seen that this is a fairly high figure for the service-price, since it would be represented by energy at 0·84*d.* with operation for eight hours a day and 300 days a year, or at 0·23*d.* for continuous operation throughout the year.

In order to indicate the general results of such calculations, the curve in Fig. 25 has been drawn out to show the conditions under which synchronous and static equipments would be of equal total cost. The curve plots the capacity of the improvement plant in quadrature kVA against service-price, so that for any cases lying above and to the left of the curve (*i.e.*, for capacities above or service-prices below those shown) the synchronous plant would be cheaper, whilst for any cases occurring below and to the right the static plant would be cheaper.

The data employed for this curve is as follows: As regards the static condensers, these have been taken at a uniform first cost of £3 2*s.* per quadrature kVA (for a fifty-period supply) with a uniform loss of $\frac{1}{2}$ kW per 100 kVA. These figures assume direct connection of the condensers without the use of transformers. The figure covering interest, depreciation and obsolescence, upkeep, rents and all other expenses resulting from the installation is 12 per cent. per annum, the corresponding figure for the synchronous condensers being 15 per cent. The difference may be taken as lying chiefly in the depreciation and upkeep, with a slight difference (particularly in smaller sizes) due to greater floor space, heavier foundations, etc.

As regards the first cost and losses for the synchronous machines, data was obtained from four different firms in this

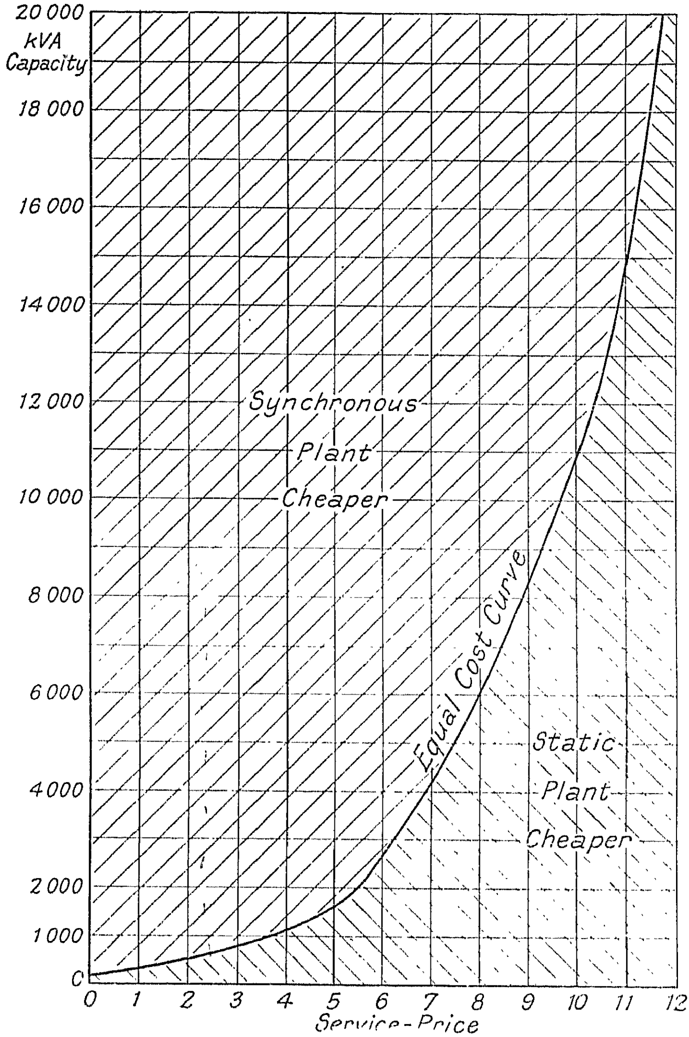

FIG. 25.—Choice of Type.

country. It was not possible to make a direct average of the four for every size, as the data did not absolutely correspond as regards voltage, speed, etc., but the figures used are a fair

average so far as this could be made, and it is believed that they are thoroughly representative of present-day prices and performances. The figures employed showed that with the annual percentages mentioned, the cost of the two types of plant, excluding losses, coincided at 193 kVA output. The curve therefore starts at this point, and for sizes below this the synchronous machine would have no economic advantages even if its losses were no greater than those of the static. Above this there will always be some service-price at which the two alternatives will cost the same, and the curve which has been drawn marks the dividing line between the spheres of utility which they respectively occupy.

Worked Examples

1. In a private generating station the capital cost of the supply plant is represented by the formula £(10 W + 5 V), where W is the kW capacity and V the kVA capacity (*i.e.*, only one-third of cost is represented by electrical gear). Capital cost of phase improvement plant is £1 10*s.* per quadrature kVA capacity. If the useful life of both types of plant is reckoned at twenty years, with zero salvage value, and if maintenance, etc., can be covered by 5 per cent. per annum on the capital cost of supply plant, or 3 per cent. on the improvement plant, what is the most economical power factor at which to work, reckoning interest at 6 per cent. per annum, and neglecting the effect of losses saved or incurred ?

Annual rate for interest and depreciation = 6 + 2·72 = 8·72 per cent.

Hence supply plant charge (proportioned to kVA) = 5 (0·0872 + 0·05) = 0·686 (£ per annum per kVA).

Improvement plant charge = $1\frac{1}{2}$ (0·0872 + 0·03) = 0·176 (£ per annum per kVA).

$$\text{Hence } \alpha = \frac{0{\cdot}176}{0{\cdot}686} = 0{\cdot}256 = \sin\phi, \text{ where } \phi = \text{most economical lag.}$$

Hence ϕ = 15° (nearly)—representing a power factor of 96·7 per cent.

2. A consumer is charged £4 per kVA on his maximum demand, the remainder of the tariff being a flat rate per kW hour, and he can install static condensers whose losses can be neglected for £3 per kVA, on which his total expenses for interest, depreciation, housing and upkeep can be reckoned at 15 per cent. per annum. Find the most economical angle of lag to which he can improve his load. If his initial power factor was 70 per cent., and his true power demand 100 kW, what will be the kVA capacity of the condensers he should install ?

$$\text{Value of } \alpha = \frac{3 \times 0{\cdot}15}{4} = 0{\cdot}1125 = \sin\phi.$$

Hence $\phi = 6\frac{1}{2}°$ and $\cos\phi$ = 99·4 per cent.

Initial p.f. = 70 per cent., hence initial lag $\phi_0 = 45\frac{1}{2}°$.

Quadrature component required = 100 ($\tan\phi_0 - \tan\phi$) = 90·7 kVA.

3. Referring to the last example, tabulate the annual costs before

and after improvement, and check the percentage saving by comparison with that found from the formula.

		£	s.	
Initial demand $= \frac{100}{0\cdot7}$	$= 142\cdot9$ kVA, which at £4 $=$	571	9	per annum
Final demand $= \frac{100}{0\cdot994}$	$= 100\cdot6$ kVA, which at £4 $=$	402	8	,,
To which must be added (quadrature component) 90·7 kVA, which at £0·45	$=$	40	16	,,
New Total . . .		443	4	,,

a saving of £128 5*s*.

Hence there is a proportionate saving of $\frac{£128\cdot2}{£571\cdot4} = 0\cdot224$, *i.e.*, $22\frac{1}{2}$ per cent.

By formula, $1 - \cos(\theta_0 - \theta) = 1 - \cos 39° = 0\cdot223$.

4. Phase improvement plant is to be installed of a total corrective capacity of 2,000 kVA, and the choice lies between static plant costing £2 15*s*. per quadrature kVA, and having a loss of 1 kW per 100 kVA, and synchronous plant costing £1 8*s*. per kVA and having a loss of 5 kW per 100 kVA. If the service averages ten hours a day (at full load) throughout the year, what must be the price of energy in order to justify the static plant, if the interest, depreciation and upkeep expenses on the two types are represented respectively by 14 per cent. and 18 per cent. per annum of the capital costs ?

Capital charge on static plant
$= 14$ per cent. of £2 15*s*. $\times$ 2,000 $=$ £770

Capital charge on synchronous plant
$= 18$ per cent. of £1 8*s*. $\times$ 2,000 $=$ £504

Difference $=$ £266

Losses charge on static plant

$$= \frac{1}{100} \times 2{,}000 \times \text{service-price} = SP \times 20$$

Losses charge on synchronous plant

$$= \frac{5}{100} \times 2{,}000 \times \text{service-price} = SP \times 100$$

Difference $= SP \times 80$

For the two alternatives to cost the same, the service-price must $= \frac{266}{80} = 3{\cdot}3$, which will correspond to a price of energy of $\frac{3{\cdot}3 \times 240}{365 \times 10} = 0{\cdot}22$ pence.

Hence the energy tariff must be $0{\cdot}22d.$ or over if the extra first cost of the static plant is to be justified.

CHAPTER XV

OTHER FACTORS

Data Employed.—The main variables in the cost of electricity supply—load and diversity factors and power factor—have now been reviewed, together with the tariffs which have been devised to cover them. It now remains to deal with a number of other factors, which, although they may be individually less important, nevertheless exercise a very great effect upon the cost. To a large extent these are interwoven with the factors already considered, and they must be regarded as other aspects of the same question, which have been grouped together into a separate chapter for the sake of convenience. The points which will here be treated are the effect of size of station and of interconnection between stations, the effect of energy price upon consumption and of consumption per head upon price. The economic progress of the past few years will also be illustrated by means of a diagram like that already employed for the two-part tariff, and the relative cost of public and private sources of supply will be touched upon.

The data which will be employed is chiefly that contained in the official publications on the subject, to which reference has already been made. In order to save frequent repetition it will be well to summarise here the main sources of this information, all of it relating to public electricity supply in this country. In connection with these figures it must be realised that a very considerable number of large firms in this country generate their own electricity. Little or no data is available regarding these private systems, but it has been estimated that in total output they may approach the sum of the public supplies. The figures employed in this book, therefore, refer only to the undertakings which are authorised to supply electricity to the public, whereas the total supply in the country is very considerably greater than this.

In 1918 the position of electricity supply in the country was investigated by the Electric Power Supply Committee under the chairmanship of Sir Archibald Williamson, and, following on the recommendations of this committee, the Electricity

Commissioners were appointed. The latter collected and published very valuable statistics, and did their best to improve the economic position, but unfortunately they were denied sufficient power to compel amalgamation or the closing down of inefficient stations, so that their work had to be chiefly of an advisory and publicity nature. Early in 1925 a committee was appointed under the chairmanship of Lord Weir, and this committee in its turn may be said to be the parent of the 1926 Act and the Electricity Board. There are now, therefore, two statutory bodies permanently in being in connection with the supply industry—the Electricity Board set up by the 1926 Act, which constitutes the governing and business authority, and the Electricity Commissioners starting a few years previously, who form the technical and advisory body.

Since their appointment the Commissioners have published annual reports, and, in addition, they have published two most valuable reports of engineering and financial statistics as shown below (a portion of the first set has already been employed in the Two-part Diagram (Fig. 14), and the other set is utilised in Fig. 27 later in this chapter) :—

Return published June, 1925, and covering three years—1920, 1921 and 1922.

Return published November, 1926, and covering two years—1923 and 1924.

(In speaking of a particular year, *e.g.*, 1924, what is meant is the twelve months ending December 31st, 1924, in the case of the companies, and March 31st or May 15th, 1925, in the case of the municipalities.) The Weir Report, some of whose findings are summarised later in the chapter, bases its figures chiefly upon the results for the year 1922 given in the earlier of these two returns.

Size of Station.—In considering the effect of size upon the cost of electricity supply, it is necessary first to distinguish between the size of the individual steam raising, prime moving and generating units, and the size of the station itself. Most of the advantages of size given below refer more particularly to large size of units, but, on the other hand, size of unit and size of station usually go together. It would clearly be inefficient to supply all the load coming on to a station by one huge unit, as this would be greatly under-loaded most of the day. Moreover there would be no effective standby plant unless a second equally large unit were installed, in which case

half the plant would be permanently idle. In practice it is usually not economical to have less than some five or six turbo-alternator sets in a station; and although these need not necessarily be all of the same size, the fact remains that for any given load factor there is usually a certain ratio between the maximum station load and the size of the largest unit which it is economical to employ. Hence in discussing the effect of size, it will be assumed that size of station and size of generating unit go hand-in-hand.

Electricity generation resembles most other forms of production in that large-scale operations are usually more economical than small. But the reasons in this case have to do less with the usual economy through sub-division of labour, etc., than with the inherent qualities of the plant employed. The effect of size can be considered under two heads:—

Effect on Standing Charges.

Large generating units are cheaper per kW than small.

If more units (as well as larger) are employed the proportion of standby plant can be diminished.

The load factor (and possibly the power factor) is likely to be somewhat better owing to the greater number and, therefore, diversity of consumers.

Effect on Working Costs.

Large units are more efficient than small.

There are also the usual labour economies of large-scale working, including the possibility of employing chemical and combustion experts, and elaborate measuring devices.

With regard to transmission and distribution, there are similar economies to be made wherever the larger scale operations are possible *within the same area* as before. But when the larger-scale working involves distribution over a larger area (the density of working remaining the same) the cost of transmission and distribution per unit is likely to be as much or more than before. Hence the chief economies of large-scale working must be looked for within the generating station itself.

Before comparing the above reasoning with the figures plotted below, it will be well to mention some of the limitations to station size. The most obvious limitation is that of the size of load. The load can be increased in either of two ways—by extending the area served (which means greater transmission and distribution costs) or by increasing the consumption of the existing area. The latter process takes time, but from com-

parisons with other countries it would appear that we are very far from having reached the saturation point as regards electricity consumption in any particular area. A second limit is imposed by the quantity of cooling water required for the condensers of a large turbine plant. This not infrequently fixes both the positions and sizes of the big stations which are to serve a particular area.

Size and Efficiency.—It is interesting to compare the above reasons for large-scale economy with the actual figures for stations of various sizes in this country. There are several difficulties in doing this, the first being the difficulty of finding a suitable basis of comparison. The best basis for this purpose is either fuel consumption per unit generated, or mean energy price charged (both being considered in relation to the kW capacity of the station), but each of these bases of comparison is open to objection. Another difficulty is that it is no use comparing a few individual stations, since conditions differ enormously—coal and water facilities, compactness and diversity of load, and, above all, station *personnel*. A small station may happen to have a first-class chief who, by giving every detail personal attention, achieves results equal to those of a far bigger station; and, conversely, a big undertaking, although able to offer a high salary, will not necessarily secure the best engineer, since neither electricity committees nor boards of directors are infallible in their judgment.

The diagram shown below plots coal consumption per unit generated, for station groups having the annual outputs shown along the base. The great objection to this basis of comparison is, of course, that it takes no account whatever of the calorific value, so that a station burning low-grade fuel will appear much less efficient than it really is. But many stations still appear to have difficulty in declaring the true average calorific value of the fuel they are using, and in most of the figures issued by the Electricity Commissioners thermal efficiencies have had to be omitted entirely. Moreover, the alternative plan of plotting *cost* of fuel per unit instead of weight is open to the objection that it takes no account of differences in situation as regards coal supplies, and therefore price.

In the annual summary of generation issued by the Commissioners, the stations are divided up as regards size of output into thirteen groups, lettered A to M. Group A consists of stations generating over 200 million units per annum,

of which there were only three in the year ending March 31st, 1925, Group B is of stations generating between 100 and 200 millions (of which there were ten), Group C generating between 50 and 100 millions, and including twenty-nine stations ; and so on down to Group M, which consists of stations generating under 50,000 units per annum. Neglecting the first and last groups as being of indeterminate size, the other groups may for approximate purposes be taken as each having an average output midway between the two limits stated, and this gives eleven groups having outputs ranging from 150 down to 0·075 million units per annum.

Under each group letter, in the summary referred to, is given the mean coal consumption per unit generated, for all the stations placed in that group. In Fig. 26 these mean consumptions are plotted to a base of the mid-size points mentioned above. The curve is in two parts, seven points being first shown to a fairly large base scale. Two points are then replotted, together with the four remaining points, in the continuation portion ; and the base measurements are now shown to one-twentieth of their previous scale, the ordinate scale being unaltered.

It will be understood that this plotting of average values in a series of groups is in many ways less satisfactory than showing actual stations, but it avoids the necessity for dealing with a large mass of figures, and gets over the difficulty of accidental individual differences. In any case it is sufficiently accurate as a guide to the general tendency, and it certainly demonstrates in a striking manner the gain in generation efficiency with size of station. A smooth curve can be drawn through all but two of the points, and it shows a steady and continuous drop in consumption as the station size increases. Moreover, the curve shows the results of only one of the gains to be anticipated from increased station size, namely, that of improved thermal efficiency. The other economic advantages which result, such as lower price of plant per kW, smaller proportion of standby plant, less labour cost, etc., and which would appear in any financial comparison such as cost per unit, do not come into the curve shown.*

As an extreme case of the economic advantages which may accrue from having a single large system in place of many small ones in serving a given area, it is instructive to compare the

* See especially a set of comparisons in *The Electrical Review* of January 1st and February 5th, 1926.

sixteen Metropolitan Companies in the London area with the single Commonwealth Edison Company of Chicago. According to figures published some years ago * the combined load factor of the London companies was 20·4 per cent. as against 45·1 per cent. for the Edison Company. The financial charges of the

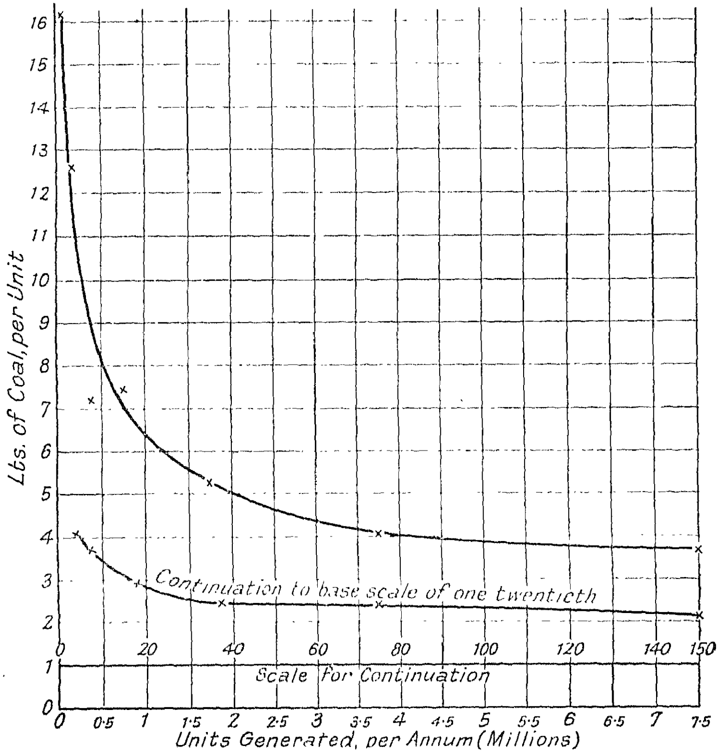

FIG. 26.—Efficiency and Station Size.

former were 50 per cent. of the total income, *i.e.*, exactly one-half of all the money received went in interest, depreciation, and amortisation. The corresponding figure for the Edison Company was only 25·4 per cent., and even this included heavy dividend payments (10·4 per cent. of the income).

Interconnection.—When it is impossible or uneconomical to concentrate the whole of generation for an area at one big

* *World Power*, September, 1924.

station it is often possible to interconnect two or more stations, and thus obtain some of the advantages of size without excessive transmission expenses. The economic advantages are generally similar to those which accompany large-sized stations, but lesser in degree, and of course the labour economies do not accrue—on the contrary, there are extra expenses involved in maintaining efficient interconnection. The chief gain is therefore that larger-sized units can be employed than would be economic in the same stations if unconnected, and that less standby plant is needed. In addition, there may be a somewhat better load factor and power factor through greater diversity of consumers.*

As regards these latter points, the advantages will be greater the greater the difference between the characteristics of the loads and stations which are interconnected. Abroad there have been some very successful examples of interconnection between stations of different types, *e.g.*, water power and coal burning, so arranged that each works under its most economical conditions, time of day, year, etc. In this country there is generally less station diversity, the great majority of stations being of the coal-steam-turbine type, so that the gain to be obtained from interconnection between stations is not so great, although very considerable.

The above remarks apply particularly to interconnection between a few stations within particular areas, a practice which has already been employed with considerable success in certain industrial areas in this country. As regards the larger question of interconnection *between* areas, this has hardly been tried here yet, but as its establishment is one of the chief aims of the 1926 Act there will doubtless soon be data by which its success in this country can be gauged. The cost of such a scheme will of course be very considerable, and it raises serious problems, such as the standardisation of frequency. But it must not be forgotten that the greater part of this country is such a compact, densely industrialised area that it represents a very favourable field for interconnection—much more so than many areas abroad which have already adopted it on a large scale.

* For a recent example see *The Electrical World*, August 6th, 1927. In the first twenty-two months' operation of the Connecticut Valley Power Exchange the net savings were £45,000. The capital invested in interconnecting transmission lines was £150,000, and this released a total generating capacity of 30,000 kW—worth four times as much as the lines cost. The resultant net saving to the three companies involved is estimated at £46,000 per annum.

Moreover, against the cost must be put all the gains enumerated above, raised to a high degree, and certain additional ones, such as the electrification of the intervening districts.

The advantages of large-scale interconnection in reducing the cost of generation in this country were given as an Appendix to the Weir Report,* and for convenience may be summarised here as follows :—

Effect on Standing Charges.

Larger generating units are cheaper per kW than small.
Smaller proportion of standby plant.
Better diversity factor (daily, because time of maximum demand varies, and annually because trade fluctuations do not coincide in all districts).
Erection of small stations avoided, whilst extensions and future needs can be met economically without undue laying down of plant before required.

Effect on Working Costs.

Large generating units are more efficient than small.
Station sites can be chosen where most economical.
Other sources of power (*e.g.*, waste heat and water) can be utilised wherever they occur.
Less efficient plant when retained can be employed for short-hour service, and the more efficient for the all-day load.

Additional Advantages.

Interconnecting mains will pick up additional loads through electrifying areas now hardly served. This country load will be different in character, and therefore improve the load factor, and in addition to the immediate economic advantages there are social and national gains in developing rural areas and decentralising industry.

Weir Report Proposals.—The figures already given for the efficiencies of different station groups, whilst serving primarily to illustrate the close connection which exists between thermal efficiency and size of station, show incidentally what a very large number of small, unconnected stations there are

* A more recent summary of the advantages of interconnection will be found in Mr. Page's Presidential Address to the Institution of Electrical Engineers, October, 1927. *Journal I.E.E.*, Vol. 66, p. 10.

in the country, and how very inefficient our supply is as a consequence. Electricity supply, like most other institutions in this country, has grown up in a spasmodic, *ad hoc* fashion, and its present position is the result of historical rather than technical factors. As in some other branches of applied science, Great Britain started by being a pioneer, and now reaps the disadvantage in a multitude of systems, voltages and frequencies, whilst countries which started later have been able to profit both by our successes and our mistakes; and by installing up-to-date systems from the first have been able to enjoy continuous, unchecked development on uniform lines. Nor can the present conditions be blamed entirely to our early start, since both "private profit" and "parish pump" politics have frequently been allowed to stand in the way of what was clearly in the public interest.

Whatever may be the historical reasons and justification, the present position is clear enough, and cannot be better expressed than in the words of the Weir Report itself :—

"The parochial policy of generation, which to-day permits "the existence of 572 authorised undertakings owning 438 "generating stations, definitely ignores the technical considera-"tions essential to cheap generation. Of the 438 generating "stations owned by authorised undertakings, not more than "about 50 can be regarded as being of really suitable size and "efficiency. Twenty-eight stations generate 50 per cent. of the "total energy, while 322 stations between them only account for "11 per cent. The percentage of standby plant is unduly high, "and the load factor is unreasonably low. Interconnection is "not carried out as a definite policy. These technical considera-"tions have been continually subordinated to other interests, "and the resultant loss to the country has been heavy, and "becomes daily heavier." * To remedy this, the Report proposed, and the 1926 Act embodied, a Central Electricity Board having power to bring about the following reforms :—

The closing down of 432 existing stations and the generation of all the electricity in twenty-eight main and thirty secondary stations, of which forty-three are existing and fifteen are to be new.

The interconnection of all these selected stations by a "grid iron" of high tension mains.

* "Report of the Committee appointed to review the National Problem of the Supply of Electrical Energy," 1926 (p. 7, par. 17).

The advantages of these two reforms have already been discussed in the previous sections. Taking all the reforms together, the Weir Report anticipated that the result should be a consumption of 500 units per annum per head of population, instead of our present figure of 100 to 200 units per head, combined with a reduction in the average overall price per unit for the country from 2*d.* to 1*d.* or less, a reduction in the proportion of standby plant from 68 per cent. to 25 per cent., an appreciable increase in the load factor, and sundry minor gains.

It is, of course, true that if the present chaotic system were allowed to continue and expand on its existing lines, it also would ultimately achieve an output of 500 units per head (though not before 1940 at the present rate of growth), and with some degree of price reduction, but the utmost that seems likely with the existing methods would be a lowering of the mean energy price from 2*d.* to 1½*d.* Under the reorganisation proposed, even if the energy price could not be reduced to less than 1*d.* a unit, the saving to the nation (as compared with 1½*d.*) would be £44,000,000 per annum.

Consumption and Price : Saturation.—That the density of consumption (*i.e.*, the consumption per head of population or per square mile of territory served) is connected with price per unit is obvious, but it is impossible to state categorically that one is the cause or consequence of the other. If the consumption density increases, costs per unit go down and the prices charged go down also. And on the other hand, if the price goes down, more energy will be sold and the consumption will increase. Each therefore reacts upon the other, and whilst it is important to realise and profit by the connection between them, there is little object in discussing which comes first.

In this connection a very interesting curve was plotted in the Weir Report, showing the relationship between the average price of supply to consumers and the number of units sold per head of population. The graph plotted was in the nature of a composite diagram, each undertaking being represented by a cross showing the average price per unit, and the number of units sold per head, in the area served by it. A smooth curve was then drawn through the mean path of these crosses, and this curve showed that where the price per unit averaged 8*d.* to 10*d.* the number of units sold per head was only 20. At 4*d.* a unit there were 50 units sold per head, at 2*d.* 140 units,

and at 1*d*. 600 units. Unquestionably, in electricity supply, price and consumption go hand-in-hand, or at least, if either can be persuaded to take the lead, the other will follow hard behind.

As regards the limits to the increase of consumption density, on innumerable occasions this country has been compared, generally very unfavourably, with other industrial countries on the score of our low electricity consumption per head. Too much should not be based on these figures, and, in particular, it must be remembered that they usually refer only to public supply; and, moreover, that as regards domestic consumption the majority of English towns can boast of an excellent gas service, having prior establishment, which largely supplements or takes the place of electricity. Nevertheless, the figures are salutary and certainly cannot be overlooked.

In the Weir Report the consumption per head from public sources is given as 110 units per annum, with the suggestion that the total might actually be 200 units per head if all private supplies could be included. Even taking the larger figure, this compares very unfavourably with countries such as the United States of America, Norway and Sweden, Switzerland, Canada, and even Tasmania, all of which show a consumption per head of population of 500 units or over. Mr. Insull, representing an area (Chicago) boasting a consumption of 1,000 units per head, anticipated that even this consumption would double in fifteen years, and gave it as his opinion that there is *no* saturation point in the demand for electricity. There is no doubt that as compared with other commodities the demand for electricity is an exceedingly elastic one, and given a sufficient reduction in price it is difficult to fix any limit to the amount of its use. Both the quantity consumed in any particular direction, and the number of directions in which it can be used, are almost inexhaustible. To mention one sphere only, at present hardly touched, which probably only awaits a sufficiently low price, the electrification of main line railways would add some 20 per cent. to the whole national demand for all other purposes, as well as most favourably affecting the national load factor.

It will be seen that whether or not any point of saturation does exist, there is no likelihood of our reaching it in this country for a very long time ahead. In the same report it was estimated that with the existing rate of increase (some 700 million units per annum, nearly 20 per cent. in the years 1922 and 1923) the consumption would reach 500 units per

head by 1940, and the Report may be said to be dominated by the aim of making it possible for this country to supply such an output in an economic fashion by that date. Whether or no the figure will be reached by then, or even before, there can be no doubt whatever that it *will* be reached, and, in fact, surpassed, in the not far distant future ; so that all plans for public electricity supply should be laid so as to make possible an output of something like five times the present one.

Three Years' Progress.—As was explained at the beginning of the chapter, there are now available, in the two statistical returns of the Electricity Commissioners, full engineering and financial statistics relating to five years of the public supply of this country and terminating December 31st, 1924, or March or May, 1925. The two-part diagram developed in an earlier chapter is very well suited for making a rapid graphical comparison between these years, and, in the figure below, the last three years whose data is available have been illustrated by means of a diagram of this sort.

The diagram itself has been fully explained in Chapter XII., and only its application to the present instance requires comment. The figures employed are summarised in the table below, and it will be seen that the first set (the ones for 1922) are the same as those on p. 213; and it will be seen also that the corresponding portion of the diagram (1922, Fig. 27) is the same as Fig. 14, except that transmission and distribution expenses are separated from generation and shown somewhat to the right. As regards the two later years, the figures are from the second return, published at the end of 1926, and the same assumptions have been made as before, except that the losses in kW are taken as 12 per cent., 10 per cent. and 10 per cent. respectively, the later figures being lower because the losses in units are shown in the return to be less. This has the effect of keeping the average consumer's load factor to 22 per cent. in all three cases, so that exactly the same scales may be employed as previously.* The figures are treated in just the same way as before, and a two-part diagram is constructed to correspond to the economic data given. This does not of course represent an "average" two-part tariff, since the data

* This assumes that the whole of the improvement in load factor, from 28 to 30 per cent. in the three years, is to the credit of the undertakings, whereas probably the individual load factors will also have risen, to a lesser extent. However, the error as compared with the assumption made above would not affect the scale sufficiently to show on the diagram.

which it averages covers domestic as well as power consumers (see note on p. 217).

	1922.	1923.	1924.
Units generated (millions)	4,500	5,230	6,020
Units sold are less by	16 per cent.	14·8 per cent.	15·4 per cent.
Fuel cost per unit generated (pence)	0·31	0·29	0·28
Fuel cost per unit sold (pence)	0·37	0·34	0·33
Aggregate demand on stations (10^6 kW)	1·83	2·075	2·298
Excess of receipts over expenses as per cent. of capital outlay	10·0 per cent.	9·65 per cent.	9·01 per cent.
Corresponding total capital charges (10^6 £s)	15·81	16·80	17·50
Total running costs (exclude fuel) (10^6 £s)	11·27	11·37	12·51
Sum of costs expressed per kW of demand	£14·78	£13·58	£13·04
Assumed loss in kW capacity during transmission and distribution	12 per cent.	10 per cent.	10 per cent.
Resulting cost to consumer per kW of demand (assuming 1·25 diversity factor)	£13·5	£12·1	£11·5

Figure 27 * shows the three years plotted together on a single diagram, and the progressive improvement at every point is clearly indicated. Whilst this improvement is quite general, it is slightly more noticeable as regards generation (G R T) than as regards transmission and distribution. The improvement is due primarily to the fact that the output has gone up without a corresponding increase in the equipment cost and

* This figure and table are reprinted from *The Electrical Review* of November 18th, 1927.

running expenses. It is also due in a slight degree to a smaller percentage surplus (here assigned to capital charges). Another point which the comparative diagram brings out very clearly is that the improvement from 1923 to 1924 was less, not only absolutely, but even proportionately, than the improvement from 1922 to 1923.

Public *v.* Private Supply.—Reference has already been made to the fact that owing to the high prices which have frequently ruled in the past, many large concerns in this country generate their own electricity for traction, colliery and other industrial purposes, thus rendering themselves independent of the public supply. Any manufacturer or private individual is perfectly free to do this, although, of course, he is not free to sell such energy to others unless authorised to do so, the latter privilege being the monopoly of the legally authorised undertaker for that district. Yet another plan is for a firm to generate its own electricity, but to maintain also a connection to the public supply for standby purposes. This, however, represents a most undesirable type of consumer to the supply authority, and usually the terms for such a connection are made virtually prohibitive.

When a small isolated firm (or a private individual) generates its own electricity this is usually because a public supply is unobtainable at that place, or only with very great difficulty. Such cases are likely to become much rarer when interconnecting mains cover most of the country. But where a large firm generates for itself this is usually because the private supply is deemed to be cheaper, and it will therefore be well briefly to consider the grounds for an economic decision of this type.

Taking extreme cases first, it will be obvious that if the firm is small and/or works at a poor load factor, the public supply is likely to be cheaper for two reasons : (1) the public authorities can employ larger and more efficient generating units ; (2) they can take advantage of the diversity factor existing between this firm and other concerns in order to improve the load factor. Where the firm in question is very large and works at a good load factor both these advantages tend to disappear, since the firm can itself utilise large sets, and there is less to be gained in the way of diversity factor with other consumers. Hence it is difficult for a supply authority to give to such a firm so great a reward for a good load factor as the firm can obtain by generating for itself.

It will be realised that the whole thing is entirely a matter of degree, and varies not only with the size and type of load, but also with the district and such things as water and coal supply, spare land available, etc. A very good summary of this type of choice is to be found in a discussion by Dr. Cramp and Mr. Julius Frith before the Manchester Association of Engineers on October 25th, 1919. The conclusions there arrived at, referring to particular types of hypothetical installation at that time, was that public supply could not compete in economy with new private plant :—

If the load factor exceeds 25 per cent. and the maximum demand exceeds 1,000 kW.

If the load factor exceeds 30 per cent. and the maximum demand exceeds 600 kW.

If the load factor exceeds 50 per cent. and the maximum demand exceeds 250 kW.

These figures could be plotted so as to show the respective economic spheres of the two types of supply in the manner of Fig. 25, but it will be understood that the actual values given above are no guide to the present-day position, and are merely to illustrate how the choice may have to be made.

Apart from the purely economic factors mentioned above, there are a number of other points not easily expressed in £ s. d., most of which are in favour of the public supply. Where there exists sufficient room in the works for a private generating station without additional land or buildings being required, it may be thought that nothing need be included for rent, rates, heating and other establishment charges, but it must not be forgotten that usually the space could be hired out or used for some other purpose. The same thing applies to staffing and supervision, including the management and general "worry," which even if they make no obvious addition to the staff pay-roll, inevitably absorb energies which could better be put into the main business of the firm. Another point is that the necessary capital may not be available, and certainly cannot be obtained so cheaply as by, say, a municipality. Other advantages of public supply are, in general, greater reliability, facilities for extensions (big or small) at any time and at short notice, and supply during off hours (nights, week-ends, etc.) for repairs or overtime work, without the expense of keeping the power house going.

APPENDICES

APPENDIX I

Summary of Equations for Interest and Sinking Funds

(These are numbered as they appear in the text, Chapter I.)

	At simple interest.	At compound interest.
Total interest I in L years due to principal P . . .	$I=PiL$ . . (1)	$I=P\{(1+i)^L-1\}$ (4)
Amount A of principal plus total interest (or future worth of present sum P) .	$A=P(1+iL)$. (2)	$A=P(1+i)^L$. (3)
		or $=P\left(1+\frac{i'}{n}\right)^{nL}$. (5)
		Alternatively, substitute from equation (6) in any of these formulæ.
The following are derived .	From equation (2)	From equation (3)
Principal or present worth which will amount to A in L years	$P=\frac{A}{1+iL}$.	$P=\frac{A}{(1+i)^L}$.
Life or length of time in which P will amount to A . .	$L=\frac{\frac{A}{P}-1}{i}$.	$L=\frac{\log\frac{A}{P}}{\log(1+i)}$.

Equivalent annually compounded interest i for an actual rate i' compounded n times per annum .

$$i=\left(1+\frac{i'}{n}\right)^n-1 \quad (6)$$

Sinking Funds accumulating at Compound Interest

(i per annum or i' compounded n times per annum)

End-of-year deposit to realise A in L years . . . . . .

$$D=\frac{Ai}{(1+i)^L-1} \quad . \ . \ . \ . \ . \quad (7)$$

Present worth of series of end-of-year deposits . . . . . .

$$P=D\frac{(1+i)^L-1}{i(1+i)^L} \quad . \ . \ . \ . \ . \quad (8)$$

Beginning-of-year deposit to realise A in L years . . . . $D = \dfrac{Ai}{(1+i)^L - 1} \times \dfrac{1}{1+i}$ . . (9)

Present worth of beginning-of-year deposits . . . . . $P = D\dfrac{(1+i)^L - 1}{i(1+i)^{L-1}}$ . . . . (10)

End-of-interval deposit (deposits made k times per annum) . . $D = A\dfrac{\left(1+\frac{i'}{n}\right)^{\frac{n}{k}} - 1}{\left(1+\frac{i'}{n}\right)^{nL} - 1}$ . . . (11)

Beginning-of-interval ditto . . $D = A\dfrac{\left(1+\frac{i'}{n}\right)^{\frac{n}{k}} - 1}{\left(1+\frac{i'}{n}\right)^{nL} - 1} \times \dfrac{1}{\left(1+\frac{i'}{n}\right)^{\frac{n}{k}}}$ (12)

APPENDIX II

Percentage Sinking Fund Deposit

(Used for estimating the cost of depreciation)

Life L.	End-of-year payment (£) to realise £100 in L years with interest compounded annually at :—					
	4 per cent.	$4\frac{1}{2}$ per cent.	5 per cent.	$5\frac{1}{2}$ per cent.	6 per cent.	7 per cent.
1	100	100	100	100	100	100
2	49·02	48·90	48·78	48·66	48·54	48·31
3	32·03	31·88	31·72	31·56	31·41	31·11
4	23·55	23·37	23·20	23·03	22·86	22·52
5	18·46	18·28	18·10	17·91	17·74	17·39
6	15·08	14·89	14·70	14·52	14·34	13·98
7	12·66	12·47	12·28	12·10	11·91	11·56
8	10·85	10·66	10·47	10·29	10·10	9·75
9	9·45	9·26	9·07	8·88	8·70	8·35
10	8·33	8·14	7·95	7·77	7·59	7·24
11	7·41	7·22	7·04	6·86	6·68	6·34
12	6·66	6·47	6·28	6·10	5·93	5·59
13	6·01	5·83	5·65	5·47	5·30	4·97
14	5·47	5·28	5·10	4·93	4·76	4·43
15	4·99	4·81	4·63	4·46	4·30	3·98
16	4·58	4·40	4·23	4·06	3·90	3·59
17	4·22	4·04	3·87	3·70	3·54	3·24
18	3·90	3·72	3·55	3·39	3·24	2·94
19	3·61	3·44	3·27	3·11	2·96	2·68
20	3·36	3·19	3·02	2·87	2·72	2·44
21	3·13	2·96	2·80	2·65	2·50	2·23
22	2·92	2·75	2·60	2·45	2·30	2·04
23	2·73	2·57	2·41	2·27	2·13	1·87
24	2·56	2·40	2·25	2·10	1·97	1·72
25	2·40	2·24	2·10	1·95	1·82	1·58

APPENDIX III

Estimated Useful Life of Engineering Assets according to Various Authorities

[In a few cases a salvage value is quoted by authorities B and C, but this never exceeds 10 per cent. of first cost, and in the present table has been omitted.]

	A.	B.	C.
Land (Freehold)	60		
Buildings (Brick or Stone)	30	50–80	50–75
,, (Wood)	—	—	25–50
Boilers, Stokers and Furnaces	—	22–25	20
Plant and Machinery	20		
Condensers	—	—	15–25
Turbines	—	—	14–20
Generators	20	30	10–20
Motors	—	25	20–30
Transformers	—	—	15–20
Underground Cables—			
H.T. Trunk Lines	40	35	20–25
Mains	25		
Overhead Conductors	25	—	20
Instruments and Meters	10	—	12–20
Storage Batteries	—	15	11–20
	D		
Track Rails	(Tram) 10–15		(Railway) 12–18*
Rolling Stock	,, 10		,, 15–30
Trolley Wire	—		2–3

A. Electricity Commissioners' suggestions for repayment periods of local authority loans, based on estimated life of the asset.
B. Engineers' Year Book, quoting Preece.
C. Standard Handbook, quoted from various American sources, including H. G. Stott, and a number of Valuation Commissions.
. Various municipal tramway undertakings.

* Depending on amount of wear.

APPENDIX IV

Efficiency of Electrical Machinery when Under-run *

In what follows it will be convenient to define the maximum or rated output of a frame in terms of fixed current and flux densities, on the understanding that this will give approximately fixed temperature rises. This point was dealt with in Chapter VIII. (p. 138), where it was shown that a large machine is characterised by a bigger efficiency than a smaller machine of the same type. It was also stated that this efficiency could, in general, be maintained above that of the smaller machine, even when the larger machine was under-run. As this question is somewhat detached from the subject-matter of any one chapter, and concerns electrical machinery as a whole, it was thought best to treat it in an appendix.

Unfortunately the simplest type of machine which can be used for illustration purposes—the "single loss" structure such as the electric cable—does not conform to either of the rules mentioned above. In this case the larger machine, running at the same current density, has just the same efficiency as the smaller one, but, on the other hand, under-running *increases* the efficiency, so that the net effect (upon which any possibility of economic choice of size depends) is the same as with more complex structures. That is to say, a larger machine used on a smaller load will have a higher efficiency than a smaller one giving its full capacity.

One of the most useful classifications of the losses in an electrical machine is as follows :—

Losses dependent upon the current and virtually independent of the pressure.

Losses dependent upon the pressure and virtually independent of the current.

Losses independent of both current and pressure.

With this grouping the total losses can be expressed as

$$c_1 I^a + c_2 E^b + c_3$$

where a, b, c are constants, I and E the line current and pressure respectively, and a and b have each very approximately the value 2. In a similar way the efficiency can be generally expressed as

$$\eta = 1 - \frac{\text{Losses}}{\text{Input}} = 1 - \frac{c_1 I^a + c_2 E^b + c_3}{EI \cos \phi}$$

* Reprinted from the *Journal I. E. E.*, 1926, Vol. 64, p. 340.

This may be illustrated by considering any rotating machine such as a D.C. shunt motor. The first group of losses, namely those represented by c_1I^a, covers the armature copper losses. These are strictly proportional to the square of the armature current with constant resistance, but the difference between armature current and line current is not great, and, moreover, this discrepancy will probably be balanced by the positive temperature coefficient of the windings.

The second group of losses, represented by c_2E^b, includes the whole of the shunt field losses and the iron losses, as can be seen by considering the effect of a change in pressure. Thus, suppose the line pressure to be halved, with the current and speed unaltered. The field losses, being equal to E^2/R, will be quartered, and the field strength, with constant permeability, will be halved. (If the permeability is not constant some field resistance must be introduced to bring the field strength down to slightly less than half what it was.) The back E.M.F. with the same speed can thus be made slightly less than half its previous value, so that with half the line pressure the difference between the applied and the back E.M.F. will be unaltered, and the same current will flow.

The flux density B is thus roughly proportional to the line pressure E, and, furthermore, it has recently been shown * that the iron losses of a D.C. machine can be most accurately assessed by coupling hysteresis and eddy losses together, and that they are very approximately proportional to B^2, that is $b = 2$ in the above formula. Finally, if c_3 represents the constant friction and windage losses, it will be seen that the whole of the losses have been accounted for.

It will be noticed that the "pressure" items are not strictly independent of current, owing to the difference between applied and back E.M.F., and for this reason the formula is most accurate when pressure and current are being varied simultaneously.

In the case of a series motor the field losses come into the c_1I^2 group, and if the pressure alone is varied the field strength must be artificially changed to correspond. In the case of machines in which part of the current is induced, this latter will not be exactly proportional to the line current, but, in spite of these various discrepancies, the above formula, with $a = b = 2$, represents most cases with a sufficient degree of accuracy for economic calculations.

On the basis of this formula it will be convenient to group electrical machines into three classes according to whether they have one, two or all three of the possible kinds of losses. There will then be :—

Simple or homogeneous machines having only one kind of loss (*i.e.*, copper or iron) ;

* *Journal I. E. E.*, 1925, Vol. 63, p. 47.

Complex static machines having two kinds of loss (copper and iron);

Complex rotary machines having all three kinds (copper, iron and frictional).

The nearest approximations in practice to machines of the first class will be found in current conductors and in iron-cored reactances. Thus in a cable (neglecting the slight dielectric losses dependent on E) the losses are proportional to I^2, and in a choking coil (neglecting the magnetising-current losses dependent on I) the losses are proportional to E^2, *i.e.*, in each case there is substantially only one class of loss.

The first point to note in a simple structure such as a cable is that the efficiency can always be raised by increasing that item of the supply upon which no losses are dependent. For, expressing the efficiency of a cable as $1 - I^2R/EI \cos \phi$, it is clear that this can be indefinitely increased by increasing E. As a rule, however, this method is inadmissible, since the pressure is fixed by the nature of the service. With regard to variations in current, the pressure remaining constant, it will be seen that the efficiency can be expressed as

$$1 - \frac{IR}{E \cos \phi} = 1 - \frac{I}{A} \cdot \frac{l\rho}{E \cos \phi}$$

where l is the length and A the cross-sectional area.

Thus, if the length is fixed, a large cable will have the same efficiency as a small one working on the same current density, but, on the other hand, if the large cable is under-run to give the smaller output, its efficiency will be increased. This shows the possibility of economic advantage to be gained from the employment of a cable larger than that physically necessary, the conditions for which are laid down in Kelvin's law.

Turning to machines of the second class, much the most important is the static transformer, the efficiency of which can be expressed as

$$\eta = 1 - \frac{c_1 I^2 + c_2 E^2}{EI \cos \phi}.$$

It is obvious from this formula that if the transformer is under-run by reducing E and I in the same ratio, the full-load efficiency will be maintained intact, provided that the power factor is not affected. Thus any particular size of transformer can be regarded, not as something capable of giving a certain output, but as something having a certain efficiency. If a higher efficiency is required, then a larger transformer must be employed. A further point is that if at full load the two groups of losses are not equal, the efficiency can actually be improved by under-running, provided this is performed in suitable proportions. Instead of considering this in detail, however, it will be more advantageous to study the more general

case given below, of which the transformer can be regarded as a particular instance.

The third class of machine (having all three varieties of loss) can be taken to include every type of rotating apparatus in which copper, iron and frictional losses occur. The efficiency can be written

$$\eta = 1 - \frac{c_1I^2 + c_2E^2 + c_3}{EI \cos \phi}$$

and it will be evident that if the line current and the line pressure are each reduced in the same ratio, the efficiency will fall owing to the presence of the constant frictional losses c_3. On the other hand, if either of the variable losses is bigger than the sum of the other two it will be possible to under-run in such a way as to reduce this larger loss and keep the efficiency as high as or even higher than on full load.

Thus suppose that the current loss is n times the sum of the other two, that is $c_1I^2 = n(c_2E^2 + c_3)$. Furthermore, suppose that it is desired to reduce the output, and therefore the input, in the ratio $1/m$ without impairing the efficiency, this reduction being accomplished purely by reducing the current. Assuming that the power factor is unaltered, the total losses must also be reduced in the same ratio ; and calling the two currents I_1 and I_2 we have :—

$$\frac{I_1}{I_2} = m = \frac{c_1I_1^2 + c_2E^2 + c_3}{c_1I_2^2 + c_2E^2 + c_3} = \frac{c_1I_1^2 + (1/n)(c_1I_1^2)}{(1/m^2)(c_1I_1^2) + (1/n)(c_1I_1^2)}$$

or $$m = \frac{1 + (1/n)}{(1/m^2) + (1/n)} = \frac{n+1}{n} \times \frac{m^2n}{n+m^2} = \frac{m^2(n+1)}{n+m^2}$$

or $$m(n+1) = n + m^2 \quad \text{or} \quad (m-1)(m-n) = 0$$

whence $$m = 1 \quad \text{or} \quad m = n.$$

The meaning of the above is that either $m = 1$ (no reduction at all) or $m = n$, *i.e.*, the output can be reduced in the ratio which the current loss bears to the sum of the other two losses. If reduced by a greater amount than this the efficiency will be lower, whilst if reduced by a less amount the efficiency will be higher than at full load. Thus in a squirrel-cage induction motor the copper losses at full load are frequently about one and one-third times the sum of the iron and frictional losses, and such a machine can be run at $1/1{\cdot}33 = 75$ per cent. of full-load current without loss of efficiency. The maximum efficiency will then occur at about 87·5 per cent. of full load and will be slightly greater than the full-load efficiency.

Exactly the same formula would apply if the losses proportional to E^2 were n times the current and frictional losses, and in this case the machine would have to be under-run by reducing the pressure while keeping the current constant. Unfortunately, in a D.C.

machine this would generally be inadmissible on account of commutation.*

If neither of the variable losses exceeds the sum of the other two, no under-running will be possible without loss of efficiency. In such a case it will be well to see what mode of under-running gives the least fall in efficiency. Putting the efficiency at full load $\eta = 1 - (c_1 I^2 + c_2 E^2 + c_3)/(EI \cos \phi)$, where I and E are the full-load line current and pressure, let I be reduced in the ratio $1/n$ and E in the ratio $1/m$, where n and m are variables but $nm =$ constant, k, the known degree of under-running. Then the new efficiency becomes

$$\eta = 1 - \frac{c_1\left(\frac{I}{n}\right)^2 + c_2\left(\frac{E}{m}\right)^2 + c_3}{\frac{E}{m} \times \frac{I}{n} \cos \phi}$$

and putting $nm = k$ or $m = \frac{k}{n}$ we get

$$\eta = 1 - \frac{c_1 I^2 n^{-2} + c_2 E^2 (n/k)^2 + c_3}{(EI/k) \cos \phi}$$

$$= 1 - \frac{k}{EI \cos \phi}\left\{c_1 I^2 n^{-2} + \frac{c_2 E^2 n^2}{k^2} + c_3\right\}$$

Assuming that $\cos \phi$ is unaffected by the under-running, this can be differentiated with respect to n, giving

$$\frac{d\eta}{dn} = 0 - \frac{k}{EI \cos \phi}\left\{-2c_1 I^2 n^{-3} + \frac{2c_2}{k^2} E^2 n + 0\right\}$$

This is zero when $\frac{c_1 I^2}{n^3} = \frac{c_2 E^2 n}{k^2}$ or $n^4 = k^2 \frac{c_1 I^2}{c_2 E^2}$

And as $n = \frac{k}{m}$, $m^4 = k^2 \frac{c_2 E^2}{c_1 I^2}$ and $\frac{n}{m} = \sqrt{\left(\frac{c_1 I^2}{c_2 E^2}\right)}$

Hence the best ratio of current reduction to pressure reduction is the square root of the ratio (full-load current losses)/(full-load pressure losses).

* No mention has been made of the case in which the indices a and b are not each equal to 2. The likelihood of this occurring is hardly sufficient to justify a detailed calculation, but in general terms it will be obvious that where the index of the variable being reduced is greater than 2, the efficiency on under-running will be better maintained than it otherwise would have been, and *vice versâ*.

APPENDIX V

Additional Questions
(No solutions given)

Part I

When is the replacement of a machine justified? A machine costs £1,000 and at the end of fifteen years is worth £150. Assuming depreciation is provided for by annually setting aside a constant percentage of the depreciated machine's value, obtain this percentage and amount in the fund at the end of ten years. (*A.M.I.M.E.*, October, 1925.)

Distinguish between maintenance and depreciation. How is the cost of the former usually met? Outline and criticise two methods of providing for the depreciation of a machine. (*A.M.I.M.E.*, October, 1926.)

Part II

State Kelvin's law for determining the most economical section of conductor for a given power transmission. If energy costs 0·75*d.* per Board of Trade unit and copper costs £70 a ton, find the best current density to use on an overhead three-phase transmission line. The energy loss on the line per annum is such that it is equal to that which would occur if full-load current flowed for 2,360 hours. The weight of copper per cubic inch is 0·315 lbs., and the specific resistance 0·67 microhms per 1-inch cube. The cost of poles and insulators may be assumed not to vary within the range of possible variation in weight of the line. Allow 12 per cent. for interest and depreciation. (*B. Sc.*, 1921.)

An average load of 10,000 kW is transmitted over a three-phase overhead line to a point at which the line pressure is 33,000 volts and the power factor 0·8. The load factor is 30 per cent., determine the most economical area of conductor, taking interest and depreciation at 10 per cent. per annum, cost of generation 0·5*d.* per unit, and cost of the complete line £$(1050A + 950)$ per mile, where A is the area of each conductor in square inches. Take the specific resistances of copper at 0·67 microhms per inch cube. (*B. Sc.*, 1926 (Internal).)

If transformers of a certain size with an average efficiency of 98·2 per cent. can be bought for 15*s.* per kVA output, and low

voltage induction motors with an efficiency of 90·5 per cent. of about the same size at 35*s.* per b.h.p., what price per b.h.p. could be paid for a high voltage induction motor for the same duty and having an efficiency of 89·5 per cent. ? Assume annual interest and depreciation of 8 per cent. for the former set and 12·5 per cent. for the latter, energy being at 1*d.* per unit and the annual load factor 30 per cent. (*B. Sc.*, 1923.)

PART III

What is meant by the terms load factor and diversity factor in central station practice ? Estimate the influence which they respectively have upon the cost of generating and distributing electrical energy. (*B. Sc.*, 1917.)

Explain the system of charging for electrical energy which depends upon the " maximum demand," and compare its advantage with (*a*) a flat rate system, and (*b*) a system in which a fixed charge per annum is made and a flat rate charge for electrical energy. (*B. Sc.*, 1924.)

Assuming that all tariffs can be expressed in the general form $p = a\,C/B + bk$, where p is the price to be charged per kW hour, C the fixed price per annum, k the cost of energy per kW hour, and B the hours of use per annum, show that p is a minimum when the ratio of the charge for energy to the fixed charge is inversely proportional to the square of the load factor expressed by B. (*B. Sc.*, 1927.)

When installing a plant for the purpose of improving the power factor of the load upon a central station, why is it not economical to instal apparatus which will make the power factor unity ? If the cost of the plant in the power station is £x per kVA, and the cost of the plant for improving the power factor is £y per kVA, find the power factor of the load which will call for the least capital expenditure. (*B. Sc.*, 1923.)

An induction motor is run for 3,000 hours in the year, at an average load of 250 h.p., at which it has a power factor of 89 per cent. and efficiency of 94 per cent. If a phase advancer costing £300 be added to raise the power factor to 0·98, find the amount of the net saving effected per annum if power costs £6 per kVA of maximum demand per annum, together with a charge of 0·6*d.* per unit. Interest and depreciation to be reckoned at 15 per cent. per annum. The sustained maximum demand is 30 per cent. in excess of the average demand, and the losses in the phase advancer are $1\frac{1}{4}$ per cent. of the average demand. (*B. Sc.*, 1926 (Internal).)

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